Chromatic symmetric functions

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Chapter 1

The Stanley- conjecture

1.1 Graphs, colorings, orientations

Definition 1.1. A *(finite simple) graph* is a pair $G = (V, E)$ where V is any (finite) set, called set of *vertices* of G, and $E \subseteq \binom{V}{2}$ is a collection of 2-subsets of V, called *edges* of G.

Definition 1.2. A *coloring* of a graph with n colors is a function $\kappa\colon V \to [n]$. A coloring is *proper* if $\{u, v\} \in E \implies \kappa(u) \neq \kappa(v)$.

Definition 1.3. We define $\chi_G(n) \coloneqq \#\{\kappa\colon V \to [n] \mid \kappa \text{ proper coloring }\}$.

Theorem 1.4 (Stanley)**.** For any graph G, $\chi_G(n)$ is a polynomial in n of degree $\#V$.

Proof. By deletion-contraction and induction on the number of edges of G.

The base case is trivial: if G has no edges, then $\chi_G(n) = n^{\#V}$. Now, let $e = \{u, v\} \in E$, let $G \setminus e$ be the graph obtained from G by removing the edge e (*deletion*), and let G/e be the graph obtained from G by identifying the endpoints of e (*contraction*). Let κ be a coloring of $G \setminus e$: if $\kappa(u) \neq \kappa(v)$, then κ is a proper coloring of $G \setminus e$ if and only if it is a proper coloring of G; if $\kappa(u) = \kappa(v)$, then κ corresponds bijectively to a coloring of G/e, and it is a proper coloring of $G \setminus e$ if and only if it is a proper coloring of G/e. This means that $\chi_{G\setminus e}(n) = \chi_G(n) + \chi_{G/e}(n)$, so

$$\chi_G(n) = \chi_{G\setminus e}(n) - \chi_{G/e}(n).$$

Now, both $G \setminus e$ and G/e have one less edge, so by induction they are polynomials in n of degree $\#V$ and $\#V - 1$ respectively. Since they have different degrees, the leading term cannot cancel, so $\chi_G(n)$ is also a polynomial in n of degree $\#V$, as desired.

At the *COmbinatorics Seminar for Everyone* we've seen a bunch of cool results about these polynomials! Let's recap.

Definition 1.5. An *orientation* of a graph is a function $\rho\colon E \to V$ such that $\rho(e) \in e$, that is, a choice of a direction for each edge of the graph. We write $u \to v$ if $\{u, v\} \in E$ and $\rho(\{u, v\}) = v$, that is, if $\{u, v\}$ is an edge of the graph directed towards v.

Definition 1.6. An orientation ρ of a graph is *acyclic* if there is no subset $\{e_1, e_2, \dots, e_k\} \subseteq E$ such that $e_{i+1} = \{\rho(e_i), \rho(e_{i+1})\}$ for all i (with $e_{k+1} \coloneqq e_1$), that is, if there is no directed loop.

Definition 1.7. A coloring κ is *(strictly) compatible* with an acyclic orientation ρ if $u \to v \implies \kappa(u) \leq \kappa(v)$ (resp. $\kappa(u) < \kappa(v)$).

A proper coloring κ induces an acyclic orientation ρ_κ, defined by $\rho_\kappa(\{u, v\}) = v$ if $\kappa(u) < \kappa(v)$ and $\rho_\kappa(\{u, v\}) = u$ otherwise. It is clear that this is the only acyclic orientation that is compatible with κ. Notice that a coloring that is strictly compatible with an orientation is automatically proper, and vice versa a proper coloring that is compatible with an orientation is automatically strictly compatible.

We immediately have the following.

Proposition 1.8.

$$\chi_G(n) = \sum_{\rho \text{ acyclic orientation of } G} \#\{\kappa \colon V \to [n] \mid \kappa \text{ proper coloring compatible with } \rho\}$$

Proof. Each coloring is compatible with exactly one acyclic orientation.

We have the following result, due to Stanley.

Theorem 1.9.

$$(-1)^{\#V} \chi_G(-n) = \sum_{\rho \text{ acyclic orientation of } G} \#\{\kappa \colon V \to [n] \mid \kappa \text{ coloring compatible with } \rho\}$$

Proof. In the proof of Theorem 1.4, we established that $\chi_G(n) = \chi_{G \setminus e}(n) - \chi_{G/e}(n)$. Multiplying by $(-1)^{\#V}$, and replacing n with $-n$, we have

$$(-1)^{\#V} \chi_G(-n) = (-1)^{\#V} \chi_{G \setminus e}(-n) + (-1)^{\#V-1} \chi_{G/e}(-n).$$

Let

$$\overline{\chi}_G(n) := \sum_{\rho \text{ acyclic orientation of } G} \#\{\kappa \colon V \to [n] \mid \kappa \text{ coloring compatible with } \rho\}.$$

We only need to show that $\overline{\chi}_G(n) = \overline{\chi}_{G \setminus e}(n) + \overline{\chi}_{G/e}(n)$.

If $\kappa(u) \neq \kappa(v)$, then the acyclic orientations of G that are compatible with κ are in bijective correspondence with the orientations of $G \setminus e$ that are compatible with κ, since given an acyclic orientation of $G \setminus e$ compatible with κ, there is only one possible way to extend it to the edge e that keeps the orientation acyclic and compatible with κ.

If $\kappa(u) = \kappa(v)$, let ρ be an acyclic orientation of G and κ a coloring that's compatible with ρ. Let $e = u \to v \in E$, and let ρ' be the orientation obtained from ρ by replacing $u \to v$ with $v \to u$. then ρ' is compatible with κ. If ρ' is also acyclic, then both the orientation induced on $G \setminus e$ and the one induced on G/e are acyclic and compatible with κ; if ρ' is not acyclic, then the orientation induced on $G \setminus e$ is acyclic and compatible with κ, and the one induced on G/e is not acyclic; this correspondence is bijective, so $\overline{\chi}_G(n) = \overline{\chi}_{G \setminus e}(n) + \overline{\chi}_{G/e}(n)$. Since the initial conditions are the same, the thesis follows.

1.2 Chromatic symmetric functions

Chromatic symmetric functions are a family of symmetric functions that generalize the concept of chromatic polynomial of a graph. We define them as follows.

Definition 1.10. Let G be a graph. We define its *chromatic symmetric polynomial* in n variables as

$$\mathcal{X}_{G,n}(x_1, \dots, x_n) = \sum_{\substack{\kappa \text{ proper} \\ n\text{-coloring}}} \prod_{v \in V} x_{\kappa(v)}.$$

Proposition 1.11. $\chi_G(n) = \mathcal{X}_{G,n}(1, 1, \dots, 1)$.

As the name suggests, the chromatic symmetric polynomial is symmetric in $x_1, \dots, x_n$. When working with symmetric polynomials, having a finite number of variables can be limiting, so it is often better to use symmetric functions instead.

We will write $X = (x_1, x_2, \dots)$ for the set of variables. Under this new framework, we can extend the definition of chromatic symmetric polynomial as follows.

Definition 1.12 ([Sta95, Definition 2.1])**.** Let G be a graph. We define its *chromatic symmetric function* as

$$\mathcal{X}_G(X) = \sum_{\substack{\kappa \text{ proper} \\ \text{coloring}}} \prod_{v \in V} x_{\kappa(v)}.$$

This is a formal power series, homogeneous of degree $\#V$, in $x_1, x_2, \dots$, that is invariant under permutation of the variables. Of course, we have the following.

Proposition 1.13. $\chi_G(n) = \mathcal{X}_G(1, 1, \dots, 1, 0, 0, \dots)$, with $x_i = 1$ for $1 \leq i \leq n$ and $x_i = 0$ for $i > n$.

A natural family of symmetric functions is given by the *elementary symmetric functions*.

Definition 1.14. For $n \in \mathbb{N}$, we define the *elementary* symmetric function of degree n as

$$e_n(X) \coloneqq \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}$$

(and $e_0 = 1$).

We recall a fundamental identity.

Proposition 1.15.

$$\prod_{i=1}^{n} (x - x_i) = \sum_{i=0}^{n} (-1)^i e_i(x_1, \dots, x_n) x^{n-i}.$$

Symmetric functions are generated, as an algebra, by the elementary symmetric functions. This means that every symmetric function can be written as a polynomial in $e_1, e_2, \dots$.

Example 1.16. The chromatic symmetric function of the graph in Figure 1.1 is $28e_4e_2 + 32e_5e_1 + 108e_6$. The graph has 168 acyclic orientations, of which 108 have one sink, and 60 have two sinks.

Indeed, we have the following.

Theorem 1.17 ([Sta95, Theorem 3.3])**.** Let G be a graph with $\#V = n$. If $\mathcal{X}_G(X) = \sum_{\lambda \vdash n} c_\lambda e_\lambda$, then the number of acyclic orientations of G with exactly k sinks is $\sum_{\lambda \vdash n, \ell(\lambda) = k} c_\lambda$.

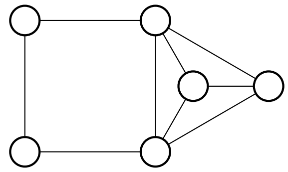


Figure 1.1: A graph.

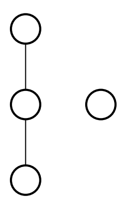
We will prove this theorem later, when we have more tools; Stanley's proof uses quasisymmetric functions, but more direct proofs exist [CS20, HJL$^+$21]. This result is an example of why positivity results are interesting: chromatic symmetric functions are easy to compute in the monomial basis of the symmetric functions (which we will define in the next chapter), and then expanding in another basis, which is easy to do, gives us combinatorial properties of the objects we are studying. We will also see how *plethysm* will give us a way to recover Theorem 1.9 from Theorem1.17.

1.3 The Stanley-Stembridge Conjecture

We now want to give the last few definitions we need to state the Stanley-Stembridge conjecture, which involves the elementary expansion of the chromatic symmetric function of certain graphs.

Definition 1.18. A (finite) *poset* is a (finite) set equipped with a partial order.

Definition 1.19. A poset $(P, \leq)$ is $3+1$-*free* if there are no $a, b, c, d \in P$ such that $a < b < c$, $a \not\leq d$, and $d \not\leq c$, i.e. if it doesn't have the pattern



Definition 1.20. The incomparability graph of a poset $(P, \leq)$ is the graph $G_P = (P, E_P)$, where $\{u, v\} \in E_P$ if and only if u and v are not comparable in P (i.e. $u \not\leq v$ and $v \not\leq u$).

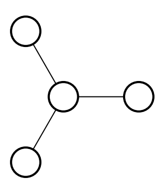


Figure 1.2: The incomparability graph of the $3+1$ poset.

Example 1.21. The chromatic symmetric function of the incomparability graph of the $3+1$ poset is $e_2{e_1}^2 - 2{e_2}^2 + 5e_3e_1 + 4e_4$.

Finally, we state the Stanley-Stembridge conjecture.

Conjecture 1.22 ([SS93, Conjecture 5.5])**.** If G is the incomparability graph of a 3+1-free poset, then $\mathcal{X}_G(X)$ is *e-positive*, that is, it is a polynomial in $e_1, e_2, \ldots$ with non-negative integer coefficients.

We have this very important result by Guay-Paquet [GP13].

Theorem 1.23. Conjecture 1.22 holds if and only if $\mathcal{X}_G(X)$ is e-positive whenever G is the incomparability graph of a $3+1$ and $2+2$-free poset.

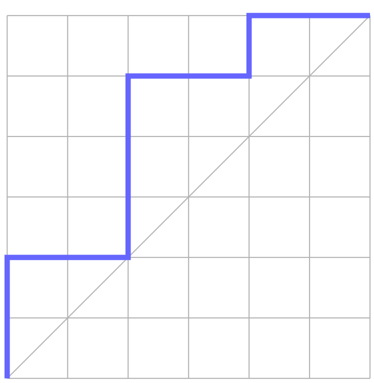
This result is a big step towards the proof of the conjecture, because $3+1$ and $2+2$-free posets are well-understood objects. Indeed, we have the following classical result.

Theorem 1.24. A poset is $3+1$ and $2+2$-free if and only if it is a *unit interval order*, that is, it is isomorphic to a collection of a length 1 (open) intervals in $\mathbb{R}$ with a partial order relation defined by $(a, a+1) \leq (b, b+1)$ if and only if $a+1 \leq b$.

In fact, being $2+2$-free is equivalent to being an interval order, and being $3+1$-free on top of that gives the unit length condition. Unit interval posets on n vertices are counted by Catalan numbers [SS58], and their incomparability graphs are easily described in terms of Dyck paths, which are lattice paths on a $n \times n$ grid, composed of north and east steps only, that lie always weakly above the diagonal $x = y$.

Theorem 1.25. There is a bijection between unit interval posets on n vertices and Dyck paths of size n.

Proof. Let P be a unit interval poset, and let $a_1 < a_2 < \cdots < a_n$ be the starting points of the intervals. Then P is isomorphic to $[n] := \{1, \dots, n\}$ with a partial order relation given by $i \leq_P j \iff a_i + 1 \leq a_j$. Identify a unit cell in the $n \times n$ grid with the coordinates of its northeastmost point and take the set of cells $C_P := \{(i, j) \in [n]^2 \mid i <_P j\}$, and take the maximal north-east path on the grid that lies below all these cells. This path is a Dyck path: indeed, if $i \geq j$ then $i \not<_P j$, so the path lies weakly above the main diagonal, and if $i \leq j$ and $j <_P k$, then $i <_P k$, so the height is increasing. Moreover, if $i \leq_P j$ then since $a_j < a_{j+1}$, we have $i \leq_P j+1$, so the cells above the path are exactly the cells in C_P, which means that the correspondence is bijective, as desired.



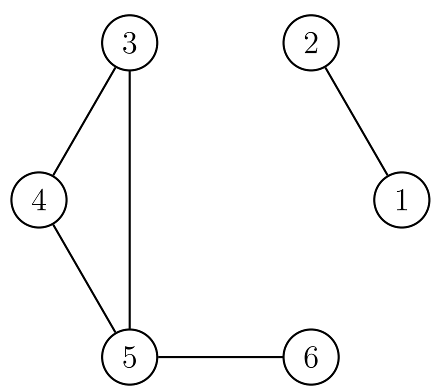


Figure 1.3: A Dyck path of size 6 and the corresponding graph.

Notice that, if G is the incomparability graph of a unit interval poset P, then for $i < j$ we have $\{i, j\} \in E \iff (i, j)$ is under the path. Given a Dyck path D, let us call G_D the graph obtained from D in this way. This means that we have the following result.

Proposition 1.26. Conjecture 1.22 holds if and only if $\mathcal{X}_{G_D}(X)$ is e-positive for any Dyck path D.

1.4 Connections with geometry

The whole theory of chromatic symmetric functions is related to geometry as well, via Hessenberg varieties.

Definition 1.27. A *Hessenberg function* is a function $h\colon [n] \to [n]$ such that $i \leq h(i) \leq h(i+1)$ for all i.

Hessenberg functions correspond naturally to Dyck paths: given a Dyck path D, if we set $h(i)$ to be the number of cells below the path in the i-th column, then h is a Hessenberg function (and vice versa). For example, the Dyck path of Figure 1.3 corresponds to the function $h = (2, 2, 5, 5, 6, 6)$. If h corresponds to D, we set $G_h \coloneqq G_D$.

Definition 1.28. Let $M \in \mathfrak{gl}_n(\mathbb{C})$, and let h be a Hessenberg function. We define the *Hessenberg variety* $\mathsf{Hess}(M, h)$ as

$$\mathsf{Hess}(M, h) \coloneqq \{V_\bullet \text{ flag} \mid MV_i \subseteq V_{h(i)}\}$$

This definition extends several common notions: for $M = 0$ or $h = (n, n, \dots, n)$ the condition $MV_i \subseteq V_{h(i)}$ is always satisfied, so we have $\mathsf{Hess}(M, h) = \mathfrak{Fl}(\mathbb{C}^n)$; for M nilpotent and $h = (1, 2, \dots, n)$ we have that $\mathsf{Hess}(M, h)$ is a Springer fiber. We have the following result (conjectured by Shareshian-Wachs, proved by Brosnan-Chow and Guay-Paquet).

Theorem 1.29. Let $S \in \mathfrak{gl}_n(\mathbb{C})$ semisimple, h Hessenberg function. Then

$$\mathsf{Frob}(H^\bullet(\mathsf{Hess}(S, h))) = \omega(\mathcal{X}_{G_h}(X, t)),$$

where Frob is the graded Frobenius character map, $H^\bullet$ is the cohomology ring of a variety, ω is the standard involution on symmetric functions, and $\mathcal{X}_{G_h}(X, t)$ is the chromatic quasisymmetric function of G_h, which gives back the chromatic symmetric function when $t = 1$.

We will give all the relevant definitions (except the definition of cohomology ring) during the rest of the course.

Chapter 2

Symmetric functions

2.1 The algebra of symmetric functions

We now formally introduce the algebra of symmetric functions.

Definition 2.1. Let $\mathbb{K}$ be a field, and consider the permutation action $S_n \circlearrowright \mathbb{K}[x_1, \dots, x_n]$ defined as $\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. The fixed points set is the subalgebra $\mathbb{K}[x_1, \dots, x_n]^{S_n}$ of *symmetric polynomials.*

We define the *symmetric functions* algebra as the inverse limit, in the category of graded rings,

$$\Lambda \coloneqq \varprojlim_n \mathbb{K}[x_1, \dots, x_n]^{S_n}$$

whose elements are formal power series of bounded degree in countably many variables that are invariant under permutation of the variables.

In the introduction, we defined the elementary symmetric functions as follows.

Definition 1.14. For $n \in \mathbb{N}$, we define the *elementary* symmetric function of degree n as

$$e_n(X) \coloneqq \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}$$

(and $e_0 = 1$).

We have two more sets of generators, defined as follows.

Definition 2.2. For $n \in \mathbb{N}$, we define the *complete homogeneous* symmetric function of degree n as

$$h_n(X) \coloneqq \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}$$

(and $h_0 = 1$).

Definition 2.3. For $n \in \mathbb{N}$, we define the *power sum* symmetric function of degree n as

$$p_n(X) \coloneqq \sum_{i=1}^{\infty} x_i^n$$

(and $p_0 = 1$).

Symmetric functions are isomorphic to the graded algebra generated by one variable in each degree. In particular, we have the following.

Proposition 2.4 ([Sta99, Theorem 7.4.4, Corollary 7.6.2, Corollary 7.7.2]).

$$\Lambda = \mathbb{K}[e_1, e_2, \dots] = \mathbb{K}[h_1, h_2, \dots] \overset{\text{char } 0}{=} \mathbb{K}[p_1, p_2, \dots]$$

With this characterization, it is natural to define the following morphism.

Definition 2.5. We define an algebra morphism $\omega \colon \Lambda \to \Lambda$ on the generators e_n by $\omega(e_n) = h_n$.

This map is central in the theory of symmetric functions. We will see that ω is an involution, an isometry, and a Hopf algebra antipode.

We now want to give some linear bases for Λ.

Definition 2.6. A *partition* $\lambda \vdash n$ of $n \in \mathbb{N}$ is an element $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathbb{N}^\omega$ such that $\lambda_i \geq \lambda_{i+1}$ for all i and $\sum \lambda_i = n$.

The *length* of a partition λ is the minimum index $\ell(\lambda)$ such that $\lambda_{\ell(\lambda)+1} = 0$.

We also define $m_i(\lambda) \coloneqq \#\{j \mid \lambda_j = i\}$, that is, the multiplicity of i in λ.

Let $\Lambda^{(n)}$ be the subspace of symmetric functions that are homogeneous of degree n. Since all the three families $\{e_n \mid n \in \mathbb{N}\}$, $\{h_n \mid n \in \mathbb{N}\}$, $\{p_n \mid n \in \mathbb{N}\}$ generate Λ as an algebra, the monomials $e_\lambda \coloneqq e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}}$ for $\lambda \vdash n$ (and the analogously defined h_λ and p_λ) generate $\Lambda^{(n)}$ as a vector space. Thus we have three bases of Λ indexed by partitions.

There are actually (at least) two more interesting bases of the symmetric functions, also indexed by partitions.

Definition 2.7. A *composition* $\alpha \vDash n$ of $n \in \mathbb{N}$ is an element $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{N}^\omega$ such that $\alpha_{i+1} > 0 \implies \alpha_i > 0$ and $\sum \alpha_i = n$.

A *weak composition* $\alpha \vDash_w n$ of $n \in \mathbb{N}$ is an element $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{N}^\omega$ such that $\sum \alpha_i = n$.

The *length* of a (weak) composition α is the number of non-zero entries in α.

For a (weak) composition $\alpha \vDash_w n$, we call $\lambda(\alpha) \vdash n$ the partition obtained from α by rearranging its entries in increasing order.

Definition 2.8. For $\lambda \vdash n$, we define the *monomial symmetric function* indexed by λ as

$$m_\lambda = \sum_{\substack{\alpha \vDash_w n \\ \lambda(\alpha) = \lambda}} {x_1}^{\alpha_1} {x_2}^{\alpha_2} \cdots$$

(and $m_0 = 1$).

In other words, m_λ is the sum of all the monomials whose exponents are exactly the parts of λ. It is clear that these elements form a basis of $\Lambda^{(n)}$.

In order to define the second basis (the *Schur functions*) we need some more definitions.

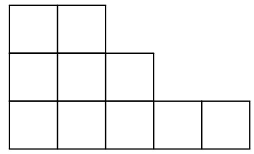


Figure 2.1: The Young diagram of $(5,3,2)$ is represented as follows.

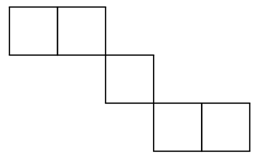


Figure 2.2: The Young diagram of the skew partition $(5,3,2)/(3,2)$

2.2 Partitions and tableaux

Definition 2.9. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, its *Young diagram* is the collection of points of the plane

$$\{(0,0),(1,0),\dots,(\lambda_1,0),(0,1),(1,1),\dots,(1,\lambda_2),\dots\}.$$

Identify each point of the diagram (i,j) with the square of vertices $(i,j),(i+1,j),(i,j+1),(i+1,j+1)$, called a *cell*. We represent a Young diagram by drawing all its cells, which results in a picture with λ_i cells in the i-th row from the bottom. Sometimes we will abuse notation and identify a partition λ with its Young diagram. See Figure 2.1 for an example.

Definition 2.10. For any partition λ, we define its *conjugate* λ' by $\lambda'_i := \#\{j \mid \lambda_j \geq i\}$, that is, the partition whose Ferrers diagram is the transpose of the one of λ.

Definition 2.11. Given two partitions λ, μ we say that $\mu \subseteq \lambda$ if the Young diagram of μ is contained in the Young diagram of λ.

Definition 2.12. A *skew partition* is any pair of partitions $\mu \subseteq \lambda$, denoted by λ/μ. Its Young diagram is the set of cells of the Young diagram of λ that are not cells of the Young diagram of μ.

See for example Figure 2.2

Definition 2.13. Given a (skew) partition ν *semi-standard young tableau* or *SSYT* is a map $T\colon \nu \to \mathbb{Z}_+$ such that when labelling each cell u of ν with $T(u)$ the resulting labelling is weakly increasing in rows and strictly increasing in columns. A SSYT is called a *standard young tableau* or *SYT* if it is a bijection $T\colon \nu \to \{1,\dots,|\nu|\}$.

For example, Figure 2.3 shows a SSYT of shape $(6,5,3,2)/(2,2,1)$.

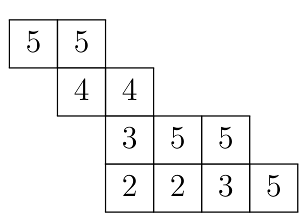


Figure 2.3: A semi-standard Young tableau of shape $(6,5,3,2)/(2,2,1)$.

2.3 Schur functions

We can finally define our last linear basis of Λ.

Definition 2.14. For a (skew) partition λ, we define the *Schur symmetric function* indexed by λ as

$$s_\lambda = \sum_{T \in \mathsf{SSYT}(\lambda)} \prod_{u \in \lambda} x_{T(u)}$$

(and $s_0 = 1$).

When λ is a partition, we have the following result.

Theorem 2.15 (Jacobi-Trudi identity)**.** For $\lambda \vdash n$, we have

$$s_\lambda = \det\left((h_{\lambda_i + j - i})_{i,j}^{\ell(\lambda)\times\ell(\lambda)} \right).$$

We endow Λ with a scalar product by declaring that the Schur functions are orthonormal.

Definition 2.16. We define the *Hall scalar product* on Λ by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.

Proposition 2.17. The morphism ω is an involution. Moreover, $\omega(p_n) = (-1)^{n-1} p_n$ and $\omega(s_\lambda) = s_{\lambda'}$, so ω is also an isometry.

Proof. First of all, let us write the generating functions for e_n, h_n, and p_n.

$$\begin{aligned}
E(t) &:= \sum_{n \in \mathbb{N}} e_n t^n = \prod_{i>0} (1 + x_i t) \\
H(t) &:= \sum_{n \in \mathbb{N}} h_n t^n = \prod_{i>0} \frac{1}{1 - x_i t} \\
P(t) &:= \sum_{n \in \mathbb{N}} p_n t^n = \sum_{i>0} \frac{1}{1 - x_i t}
\end{aligned}$$

We have the identities

$$\begin{aligned}
t\frac{d}{dt} H(t) &= \left(\sum_{i>0} \frac{x_i t}{1 - x_i t} \right) \left(\prod_{i>0} \frac{1}{1 - x_i t} \right) &= (P(t) - 1) H(t) \\
t\frac{d}{dt} E(t) &= \left(\sum_{i>0} \frac{x_i t}{1 + x_i t} \right) \left(\prod_{i>0} (1 + x_i t) \right) &= (1 - P(-t)) E(t).
\end{aligned}$$

Equating the coefficients, we have

$$n h_n = \sum_{i=1}^{n} p_i h_{n-i} \quad \text{and} \quad n e_n = \sum_{i=1}^{n} (-1)^{i-1} p_i e_{n-i}.$$

Applying ω to the second identity and equating the coefficients again, we get $\omega(p_n) = (-1)^{n-1} p_n$, as desired. This implies that ω is an involution.

Now, notice that $H(t)E(-t) = 1$. We have the identity

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = \delta_{n,0}$$

which is equivalent to saying that, given $n \in \mathbb{N}$, the matrices $(h_{i-j})_{i,j}^{n\times n}$ and $((-1)^{i-j}e_{i-j})_{i,j}^{n\times n}$ are inverses of each other. If λ is any partition such that $\ell(\lambda)+\ell(\lambda') \leq n$, looking at the appropriate minors one can show that

$$\det\left((h_{\lambda_i+j-i})_{i,j}^{\ell(\lambda)\times\ell(\lambda)}\right) = \det\left((e_{\lambda'_i+j-i})_{i,j}^{\ell(\lambda)\times\ell(\lambda)}\right).$$

Applying ω and recalling Theorem 2.15, we have $\omega(s_\lambda) = s_{\lambda'}$, as desired.

Let us show the following characterization of skew Schur functions.

Proposition 2.18. For any triple of partitions λ, μ, ν with $\mu \subseteq \nu$ we have $\langle s_\mu s_\nu, s_\lambda\rangle = \langle s_{\lambda/\mu}, s_\nu\rangle$.

Proof. Define the coefficients $c_{\mu\nu}^{\lambda}$ and $\tilde{c}_{\mu\nu}^{\lambda}$ via the expansions

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda \qquad\qquad s_{\lambda/\mu} = \sum_\nu \tilde{c}_{\mu\nu}^\lambda s_\nu.$$

Consider three alphabets of variables X, Y and Z. Then we have

$$\begin{aligned}
\sum_{\lambda,\mu,\nu} \tilde{c}_{\mu\nu}^\lambda s_\mu[X]s_\nu[Y]s_\lambda[Z] &= \sum_{\lambda,\mu} s_\mu[X]s_{\lambda/\mu}[Y]s_\lambda[Z] \\
&= \sum_\lambda s_\lambda[X+Y]s_\lambda[Z] \\
&= \sum_{n\in\mathbb{N}} h_n[(X+Y)Z] = \sum_{n\in\mathbb{N}} h_n[XZ+YZ] \\
&= \sum_{n\in\mathbb{N}}\sum_{i=1}^{n} h_i[XZ]h_{n-i}[YZ] \\
&= \left(\sum_{n\in\mathbb{N}} h_n[XZ]\right)\left(\sum_{m\in\mathbb{N}} h_n[YZ]\right) \\
&= \left(\sum_{\mu} s_\mu[X]s_\mu[Z]\right)\left(\sum_{\nu} s_\nu[Y]s_\nu[Z]\right) \\
&= \sum_{\mu,\nu} s_\mu[X]s_\mu[Z]s_\nu[Z]s_\nu[Y] \\
&= \sum_{\mu,\nu,\lambda} c_{\mu\nu}^\lambda s_\mu[X]s_\lambda[Z]s_\nu[Y].
\end{aligned}$$

Corollary 2.19. For all skew partitions we have $\omega s_{\lambda/\mu} = s_{(\lambda/\mu)'}$.

Proof. For all partitions ν we have, by Proposition 2.18 and the fact that ω is an isometry we have

$$\langle \omega s_{\lambda/\mu}, s_\nu\rangle = \langle s_{\lambda/\mu}, s_{\nu'}\rangle = \langle s_\mu s_{\nu'}, s_\lambda\rangle = \langle s_{\mu'} s_\nu, s_{\lambda'}\rangle = \langle s_{\lambda'/\mu'}, s_\nu\rangle$$

2.4 Representations of the symmetric group

We quickly recall some notions about representations of groups.

Definition 2.20. A *representation* of a group G is a (complex) vector space V equipped with a linear map $G \to GL(V)$ (that is, an action of G on V). A representation is *irreducible* if $V = X \oplus Y$ with X, Y representations implies $X = \{0\}$ or $Y = \{0\}$.

Definition 2.21. Given a representation V, we define its *character* as the map $\chi_V : G \to \mathbb{C}$ defined by $\chi_V(g) = \mathsf{trace}(g)$.

Note that a (complex) representation of a finite group is determined (up to isomorphism) by its character, so we often identify the two with an abuse of notation. Moreover, we have a correspondence between characters of the symmetric group and symmetric functions, as follows.

Definition 2.22. We define the *Frobenius characteristic* of a representation V of S_n as

$$\mathsf{Frob}(V) := \frac{1}{n!} \sum_{\sigma \in S_n} \chi_V(\sigma) p_{c(\sigma)}$$

where $c(\sigma)$ is the *cycle type* of σ, that is, the partition of n such that the sizes of its blocks are the lengths of the cycles of σ.

Since the trace is invariant under conjugation, the character is constant on the conjugacy classes, which are determined by the cycle type; we can thus define $\chi_V(\mu) := \chi_V(\sigma)$ where σ is any permutation with cycle type μ. By grouping the elements with the same cycle type, we have

$$\mathsf{Frob}(V) = \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_V(\mu) p_\mu$$

where $z_\mu = \prod_i i^{m_i(\mu)} m_i(\mu)!$, which is the size of the centralizer of a permutation σ with cycle type μ.

If G is a finite group, we have a natural scalar product defined on the algebra $\mathsf{CF}(G)$ of class functions of a representation (which is the center of $\mathbb{C}[G]$), as follows.

Definition 2.23. Let χ, ψ be two class functions of G. We define the scalar product

$$\langle \chi, \psi \rangle := \frac{1}{\#G} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

With this definition, characters of irreducible representation form an orthonormal basis. When $G = S_n$, remembering the identification between representations of a group and their characters, we have the following.

Theorem 2.24. The map $\mathsf{Frob} \colon \bigoplus_{n \in \mathbb{N}} \mathsf{CF}(S_n) \to \Lambda$ is a bijective isometry. Moreover, irreducible representations of S_n corresponds to Schur functions and vice versa, and if χ, ψ are characters of S_m and S_n respectively, then

$$\mathsf{Frob}\left(\mathsf{Ind}_{S_m \times S_n}^{S_{m+n}}(\chi\psi)\right) = \mathsf{Frob}(\chi)\mathsf{Frob}(\psi).$$

We call V_λ the irreducible representation whose character χ_λ corresponds to the Schur function s_λ. Since the characters of irreducible representations are orthonormal, the coefficient of s_λ in $\mathsf{Frob}(V)$ is the multiplicity of V_λ in V.

If V is graded, we can refine the definition of Frobenius characteristic as follows.

Definition 2.25. We define the *graded Frobenius characteristic* of a representation $V = \bigoplus_{d\in\mathbb{N}} V_d$ as

$$\mathsf{Frob}(V)(t) = \sum_{d\in\mathbb{N}} \mathsf{Frob}(V_d)t^d \in \Lambda[[t]].$$

2.5 Extra structure

Λ is a Hopf algebra and a λ-ring. We'll see more about this later in the course.

2.6 Resources

For more about symmetric functions, see [Ale].

Chapter 3

Quasisymmetric functions

Sometimes combinatorial objects are naturally associated with formal power series that are not symmetric functions, but that are still invariant in some sense. Quasisymmetric functions appear very often in combinatorics, and they are a useful tool to keep in mind even when working with symmetric functions.

Definition 3.1. Let $\mathbb{K}$ be a field, and let

$$\mathbb{K}[x_1, x_2, \dots] := \varprojlim_n \mathbb{K}[x_1, \dots, x_n]$$

(where the limit is again taken in the category of graded rings) be the ring of formal power series of bounded degree in countably many variables. We say that $F \in \mathbb{K}[x_1, x_2, \dots]$ is *quasisymmetric* if, for any $\alpha \vDash n$ of length ℓ, the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}$ in F is the same as the coefficient of $x_{f(1)}^{\alpha_1} x_{f(2)}^{\alpha_2} \cdots x_{f(\ell)}^{\alpha_\ell}$ for any strictly increasing function $f\colon \mathbb{Z}_+ \to \mathbb{Z}_+$.

We call QSym the ring of quasisymmetric functions. There is a very natural basis on QSym, indexed by compositions.

Definition 3.2. For $\alpha \vDash n$ of length ℓ, we define the *monomial quasisymmetric function* indexed by α as

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_\ell} {x_{i_1}}^{\alpha_1} {x_{i_2}}^{\alpha_2} \cdots {x_{i_\ell}}^{\alpha_\ell}$$

(and $M_0 = 1$).

Clearly, $m_\lambda = \sum_{\lambda(\alpha)=\lambda} M_\alpha$.

3.1 Fundamental quasisymmetric functions

There is another basis, introduced by Gessel, which is extremely relevant in this branch of algebraic combinatorics.

Definition 3.3. For $\alpha \vDash n$ of length ℓ, we define the *(Gessel) fundamental quasisymmetric function* indexed by α as

$$F_\alpha = \sum_{\beta \leq \alpha} M_\beta,$$

where $\leq$ is the refinement order (and $F_0 = 1$).

Since the base change matrix between M and F is unitriangular, the fundamental quasisymmetric functions form a basis of QSym. There is an equivalent definition, in term of subsets.

Definition 3.4. For $S \subseteq [n-1]$, we define the *(Gessel) fundamental quasisymmetric function* of degree n indexed by S as

$$F_{n,S} = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ j \in S \implies i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

It is easy to see that, for $\alpha \vDash n$, $F_\alpha = F_{n,S_\alpha}$, where $S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}\}$. When the degree is clear from the context, we sometimes write F_S for $F_{n,S}$.

The fundamental basis is nice for several reasons, one of which is that fundamental functions are easy to multiply. We have the following.

Theorem 3.5. Let $\sigma \in S_m$, $\tau \in S_n$, and let $\sigma \sqcup \tau$ be the set of elements of S_{m+n} such that, in one-line notation, the elements $\{1, \ldots, m\}$ read σ and the elements $\{m+1, \ldots, m+n\}$ read τ shifted up by m. For $\sigma \in S_n$, let $\mathsf{Des}(\sigma) = \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}$. Then

$$F_{m,\mathsf{Des}(\sigma)} F_{n,\mathsf{Des}(\tau)} = \sum_{\pi \in \sigma \sqcup \tau} F_{m+n,\mathsf{Des}(\pi)}.$$

Notice that each subset of $[n-1]$ can be obtained as descent set of a permutation, so the theorem gives a rule to multiply any two fundamental quasisymmetric functions, and that the multiplication rule is positive (that is, the fundamental expansion of the product of two fundamental quasisymmetric functions has positive integer coefficients).

Another reason is that we can use them to derive an extension of ω to QSym; actually, there are two such extensions.

Definition 3.6. We define $\omega\colon \mathsf{QSym} \to \mathsf{QSym}$ and $\psi\colon \mathsf{QSym} \to \mathsf{QSym}$ as

$$\omega(F_{n,S}) \coloneqq F_{n,{}^c(n-S)} \qquad \text{and} \qquad \psi(F_{n,S}) \coloneqq F_{n,n-S},$$

where $n - S \coloneqq \{n - i \mid i \in S\}$, and ${}^c(n-S)$ is its complement.

Both ω and ψ coincide with the previously defined ω on symmetric functions: we can see that from the fundamental expansion of a Schur function.

Definition 3.7. For $T \in \mathsf{SYT}(\lambda/\mu)$ and $a \in [|\lambda/\mu| - 1]$, we say that a is a *descent* of T if $T(u) = a$ and $T(v) = a + 1$, then the cell v is to the right of the cell u in the Young diagram of λ.

Proposition 3.8 ([Sta99, Theorem 7.19.7])**.** For any skew partition λ/μ, we have

$$s_{\lambda/\mu} = \sum_{T \in \mathsf{SYT}(\lambda/\mu)} F_{\mathsf{Des}(T)}$$

Proof. Via standardization, check the references.

Using the Jacobi-Trudi identity, we can extend the definition of Schur functions to compositions, in the following way.

Definition 3.9. For $\alpha \vDash n$, we define

$$s_\alpha := \det\left((h_{\alpha_i+j-i})_{i,j}^{\ell(\alpha)\times\ell(\alpha)} \right).$$

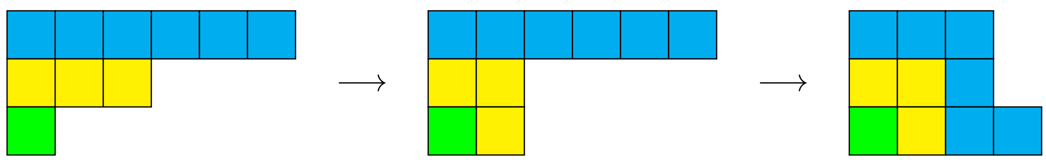
Notice that swapping two rows of the matrix corresponds to swapping two parts of α, up to changing them by one unit. Namely, if $\alpha = (\alpha_1, \dots, \alpha_\ell)$, let

$$\sigma_i(\alpha) = (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1} - 1, \alpha_i + 1, \alpha_{i+2}, \dots, \alpha_\ell).$$

Then $s_\alpha = -s_{\sigma_i(\alpha)}$. This can be represented graphically via the *slinky rule*, of which we give an intuitive formulation.

Proposition 3.10 (Slinky rule [ELW10])**.** Let $\alpha \vDash n$. Draw the Ferrers diagram of α, then bend the rows downwards until you obtain a partition λ. If you can't, then $s_\alpha = 0$; otherwise $s_\alpha = \pm s_\lambda$.

Example 3.11. For example, if $\alpha = (1, 3, 6)$, then $\lambda = (4, 3, 3)$.



Proposition 3.12 ([Sta99, Ges19])**.** Let $f \in \Lambda$. Symmetric functions are quasisymmetric, so f admits a fundamental basis expansion. Then

$$f = \sum_{\alpha \vDash n} c_\alpha F_\alpha \implies f = \sum_{\alpha \vDash n} c_\alpha s_\alpha.$$

Proof. Check the references. The nicest proof is the one in [Ges19], via a sign-reversing involution on SYTs.

3.2 Posets

We can associate a quasisymmetric function to a poset in the following way.

Definition 3.13. Let P be a finite poset. We define the quasisymmetric function associated to P as

$$\mathcal{X}_P(X) := \sum_{\substack{\kappa\colon P \to \mathbb{Z}_+ \\ \kappa \text{ strictly increasing}}} \prod_{v \in P} x_{\kappa(p)}.$$

We can derive the fundamental basis expansion via linear extensions.

Definition 3.14. A *linear extension* of a poset P with n elements is an order-preserving bijection $\psi\colon P \to [n]$.

Proposition 3.15. Let ψ be a linear extension of P^*, i.e. an order-reversing bijection $\psi\colon P \to [n]$. For α linear extension of P, define $\sigma_\psi(\alpha) = (\psi(\alpha^{-1}(i)))_{1 \le i \le n} \in S_n$. We have

$$\mathcal{X}_P(X) = \sum_{\substack{\alpha \text{ linear} \\ \text{extension}}} F_{\mathsf{Des}(\sigma_\psi(\alpha))}.$$

This little result is going to be useful later.

Proposition 3.16. Let $\varphi\colon \mathsf{QSym} \to \mathbb{Q}[q]$ defined on the fundamental basis as $\varphi(F_{n,S}) = q(q-1)^i$ if $S = \{i+1, i+2, \dots, n-1\}$ and 0 otherwise. Then, if P is a poset with n elements, $\varphi(\mathcal{X}_P(X)) = q^m$, where m is the number of minimal elements of P.

proposition

Proof. Let $\psi\colon P \to [n]$ be an order-reversing bijection. Since ψ is order-reversing, the only way to obtain a linear extension α with descent set $\{i+1, i+2, \dots, n-1\}$ is as follows: let v be the minimal element of P for which $\psi(v)$ is greatest, then choose any i minimal elements of P other than v, list them in increasing order according to ψ, then list v, then list the remaining elements of P in decreasing order according to ψ. Since there are $\binom{m-1}{i}$ choices for the minimal elements, we have

$$\varphi(\mathcal{X}_P(X)) = \sum_{i=0}^{m-1} q(q-1)^i = q \sum_{i=0}^{m-1} (q-1)^i 1^{m-1-i} = t^m$$

as desired.

3.3 Hecke algebras

In a representation theoretical sense, fundamental quasisymmetric functions play for 0-Hecke algebras the same role that Schur functions play for the symmetric group.

Definition 3.17. A (type A) *Hecke algebra* is an algebra $\mathcal{H}_q$ generated over $\mathbb{C}(q)$ by the elements $T_1, \dots, T_{n-1}$ satisfying the relations:

1. $(T_i - q)(T_i + 1) = 0$;
2. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$;
3. if $|i-j| > 1$, then $T_i T_j = T_j T_i$.

If we set $q = 1$, it's immediate that $\mathcal{H}_1 = \mathbb{C}[S_n]$. Recall that representations of S_n are the same thing as $\mathbb{C}[S_n]$-modules, and that irreducible ones are in correspondence, via the Frobenius characteristic, with Schur functions.

We can define an analogous map for $\mathcal{H}_0$ modules: irreducible modules are indexed by compositions $\alpha \vDash n$, and if we call F^α the module indexed by α, we can define $\mathsf{Frob}(F^\alpha) \coloneqq F_\alpha$, and this map is an algebra isomorphism between the algebra of representations of $\mathcal{H}_0$ (with the "induction" product) and QSym.

Chapter 4

Chromatic quasisymmetric functions

This chapter is based on [SW14].

In Chapter 1 we defined the chromatic symmetric function.

Definition 1.12 ([Sta95, Definition 2.1])**.** Let G be a graph. We define its *chromatic symmetric function* as

$$\mathcal{X}_G(X) = \sum_{\substack{\kappa \text{ proper} \\ \text{coloring}}} \prod_{v \in V} x_{\kappa(v)}.$$

And we stated the famous conjecture.

Conjecture 1.22 ([SS93, Conjecture 5.5])**.** If G is the incomparability graph of a 3+1-free poset, then $\mathcal{X}_G(X)$ is *e-positive*, that is, it is a polynomial in $e_1, e_2, \ldots$ with non-negative integer coefficients.

The following notation will make the formulas in this chapter more succinct.

Notation 4.1. We denote by $\mathcal{PC}(G)$ the set of proper colorings of G.

Notation 4.2. For κ any coloring of G, set $x^\kappa = \prod_{v \in V} x_{\kappa(v)}$.

Thus we can rewrite:

$$\mathcal{X}_G(X) = \sum_{\kappa \in \mathcal{PC}(G)} x^\kappa.$$

We now define the a generalisation called the chromatic *quasi*-symmetric function.

Definition 4.3. Let $G = (V, E)$ be a simple graph and suppose V is a finite set of positive integers. Let $\kappa : V \to \mathbb{P}$ be a coloring of G. Then $(i, j) \in V \times V$ is called and *ascent* of κ if $i < j$ and $\kappa(i) < \kappa(j)$ and a *descent*. The number of ascents and descents of κ are denoted by $\mathsf{asc}(\kappa)$ and $\mathsf{des}(\kappa)$, respectively.

Definition 4.4. Let $G = (V, E)$ be a graph with $V \subseteq \mathbb{P}$. We define its *chromatic quasisymmetric function* as

$$\mathcal{X}_G(X, t) = \sum_{\kappa \in \mathcal{PC}(G)} t^{\mathsf{asc}(\kappa)} x^\kappa.$$

Clearly $\mathcal{X}_G(X, 1) = \mathcal{X}_G(X)$. It is easy to see that for all G, $\mathcal{X}_G(X, t)$ is a quasisymmetric function with coefficients in $\mathbb{Z}[t]$. The following example illustrates that $\mathcal{X}_G(X, t)$ is not always a symmetric function, and depends not only on the isomorphism type of the graph.

Figure 4.1

Example 4.5. Consider the labelled graphs in Figure 4.1. If G is the labelled graph on the left we have that $\mathcal{X}_G(X,t)$ is symmetric and equal to $tm_{2,1} + (t^2+4t+1)m_{1,1,1}$. If G is the labelled graph on the right, then $\mathcal{X}_G(X,t)$ is not symmetric and equal to $tM_{2,1} + M_{1,2} + (t^2+4t+1)M_{1,1,1}$.

If $\mathcal{X}_G(X,t)$ is a symmetric is easy to see that it is equivalent to count ascents or descents, indeed for any proper coloring κ with maximal color m, we can obtain proper coloring κ' by replacing each color i with $m-i+1$ and $\mathsf{asc}(\kappa) = \mathsf{des}(\kappa')$. So we have the following.

Proposition 4.6. If $X_G(X,t) \in \Lambda_{\mathbb{Z}[t]}$, then

$$\mathcal{X}_G(X,t) = \sum_{\substack{\kappa \text{ proper} \\ \text{coloring}}} t^{\mathsf{des}(\kappa)} \prod_{v \in V} x_{\kappa(v)}.$$

Conjecture 4.7 ([SW14])**.** If G is the incomparability graph of a unit interval order, then $\mathcal{X}_G(X,t)$ is e-positive.

Example 4.8. If $G = ([n], \emptyset)$ is the empty graph,

$$\mathcal{X}_G(X,t) = e_1^n.$$

Example 4.9. If $G = \left([n], \binom{[n]}{2}\right)$ is the complete graph then

$$\mathcal{X}_G(X,t) = e_n \cdot \sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{inv}(\sigma)} = [n]_t! e_n,$$

where the second equally is a well-known fact established by MacMahon.

Theorem 4.10. If G is the incomparibility graph of a unit interval order then $\mathcal{X}_G(X,t)$ is a symmetric function.

The proof of this fact relies on the following lemma.

Lemma 4.11. Take G to be the incomparability graph of a unit interval order and κ a proper coloring of G. For $a \in \mathbb{N}$, set $G_{\kappa,a}$ to be the sub-graph of G formed by the vertices colored a or $a+1$. Then the connected components of $G_{\kappa,a}$ are chains of the form $i_1 — i_2 — \cdots — i_k$ with $i_1 < \cdots < i_k$.

Proof. Set $G_{\kappa,a} = (V, E)$. Take $x, y, z \in V$ with $\{x, y\}, \{y, z\} \in E$. It is easy to see from the Dyck path representation of G that if $x < y > z$ or $x > y < z$, we must have $\{x, z\} \in E$. But since $G_{\kappa,a}$ is bipartite it cannot have a 3-cycle. So we must either have $x < y < z$ or $x > y > z$. This implies that any longer chain must also have increasing/decreasing labels and so there are no cycles in $G_{\kappa,a}$. Now it suffices to show that there are no vertices of degree ≥ 3. Suppose x has at least 3 neighbors a, b, c. Then there are three chains: $a — x — b$, $a — x — c$ and $c — x — b$ and it is impossible for all three of them to have increasing labels.

Proof of Theorem 4.10. We will define a map ϕ_a on the set of proper colorings of G that exchanges the number of occurrences of the color a with the number of occurrences of the color $a + 1$ and conserves the asc-statistic. This implies the symmetry of $\mathcal{X}_G(X, t)$.

Consider the sub-graph $G_{\kappa,a}$ as in Lemma 4.11. For all its chains of odd length, change the colors from a to $a + 1$ and vice versa. Define $\phi_a(\kappa)$ to be the resulting coloring.

4.1 Schur expansion

In this section we will show the Schur positivity of $\mathcal{X}_G(X, t)$, for G the incomparability graph of a unit interval order. The argument is a generalisation of Gasharov's for $\mathcal{X}_G(X, 1)$ [Gas96] and can be found in [SW14].

Definition 4.12. Take P a poset of size n and λ a partition of n. A *P-tableau of shape* λ is a filling of a Young diagram (French notation) of shape λ such that

(i) each element of P appears exactly once;

(ii) the rows are increasing in P from left to right;

(iii) the columns are not decreasing in P from bottom to top. In other words, if y sits on top of x, we have either $x <_P y$ or x and y are incomparable.

Denote by $\mathcal{T}_{P,\lambda}$ the set of such P-tableau of shape λ.

Definition 4.13. Given A a P-tableau and G the incomparibility graph of P, an edge $\{i, j\}$ is said to be a G *inversion* of A of $i < j$ and the row of i is above the row of j in A. The number of G-inversions of A is denoted by $\mathsf{inv}_G(A)$.

For an example of a P-tableau and its inversions, see Figure 4.2.

Theorem 4.14. If G is the incomparability graph of a unit interval order P of size n then

$$\mathcal{X}_G(X, t) = \sum_{\lambda \vdash n} \left(\sum_{T \in \mathcal{T}_{P,\lambda}} t^{\mathsf{inv}_G(T)} \right) s_\lambda$$

Example 4.15. Take $P_{n,r}$ to be the poset on $[n]$ defined by $i <_{P_{n,r}} j$ if $j - i \geq r$. In Figure 4.3, we have drawn all the $P_{3,2}$-tableaux. Theorem 4.14 states that for G the incomparability graph of $P_{3,2}$ we have

$$\mathcal{X}_G(X, t) = ts_{(2,1)} + (t^2 + 2t + 1)s_{(1,1,1)}.$$

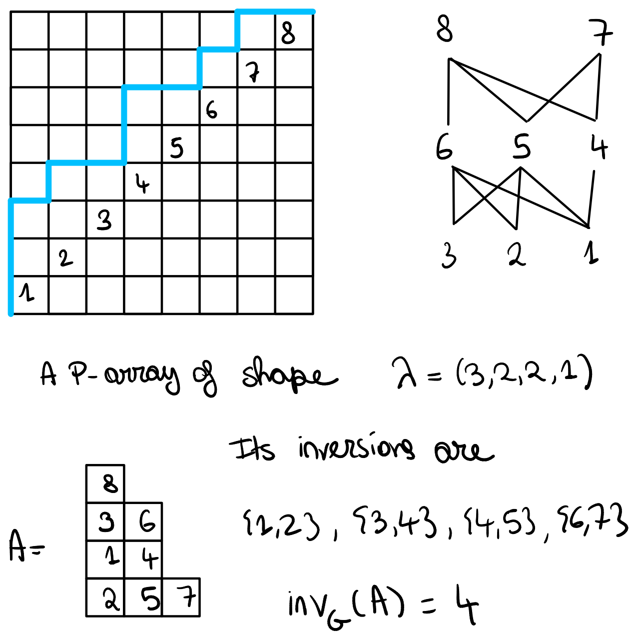


Figure 4.2: A P-tableau of a unit interval order and its G-inversions.

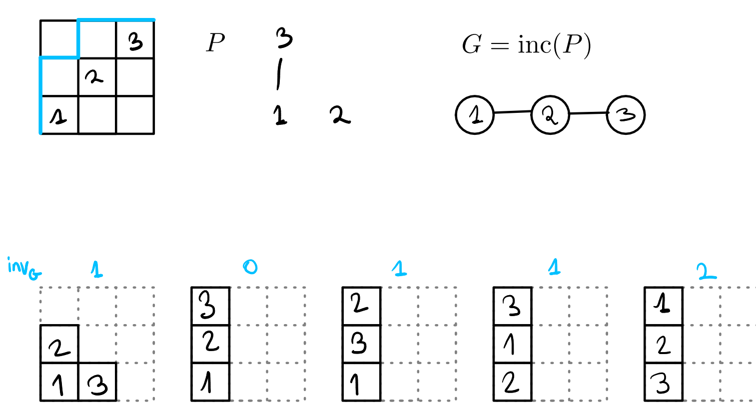


Figure 4.3: $P_{3,2}$ and its tableaux

For this proof, we will need a more general notion than P-tableaux.

Definition 4.16. Let P be a poset of size n and $\alpha = (\alpha_1, \dots, \alpha_n)$ a weak composition of n. A P-array is a filling of the n^n diagram with elements of P such that each element of P appears exactly once and the filling of the i-th row from the bottom is of the form $i_1, i_2, \cdots, i_{\alpha_i}, 0, \dots, 0$ with $i_1 <_p \cdots <_p i_{\alpha_k}$. Define $\mathsf{inv}_G(A)$ of a P-array in the same way as for P-tableaux: the number of edges $\{i, j\}$ of G with $i < j$ and i in a row above j in A.

Proof of Theorem 4.14. Let $c_\lambda(t)$ be the coefficients defined by

$$\mathcal{X}_G(X, t) = \sum_{\lambda \vdash n} c_\lambda(t) s_\lambda.$$

We have to show that, for any $\lambda \vdash n$,

$$c_\lambda(t) = \sum_{T \in \mathcal{T}_{P,\lambda}} t^{\mathsf{inv}_G(T)}. \tag{4.1}$$

Let $\overline{\lambda}$ be the weak composition obtained from λ by adding zeroes so that it has length n (if necessary).

For any integer vector α we set $h_\alpha = \prod_i h_{\alpha_i}$, using the conventions $h_0 = 1$ and $h_m = 0$ whenever $m < 0$.

By the definition of the Hall scalar product and the Jacobi-Trudi identity (Theorem 2.15)

$$\begin{aligned} c_\lambda(t) &= \langle \mathcal{X}_G(X, t), s_\lambda \rangle \\ &= \left\langle \mathcal{X}_G(X, t), \det((h_{\lambda_i - i + j})_{i,j}^{\ell(\lambda) \times \ell(\lambda)}) \right\rangle \\ &= \left\langle \mathcal{X}_G(X, t), \det((h_{\overline{\lambda}_i - i + j})_{i,j}^{n \times n}) \right\rangle \end{aligned}$$

For any $\sigma \in \mathfrak{S}_n$ define $\delta_\sigma(\lambda) \in \mathbb{Z}^n$ such that

$$\delta_\sigma(\lambda)_i \coloneqq \overline{\lambda}_{\sigma(i)} - \sigma(i) + i$$

so that we have (by definition of the determinant)

$$c_\lambda(t) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \left\langle \mathcal{X}_G(X, t), h_{\delta_\sigma(\lambda)} \right\rangle$$

For any integer vector α, set $\mathcal{PC}_\alpha(G)$ to be the set of proper colorings of G with α_i occurrences of the letter i (the set is empty when a component of α is negative). Using Proposition 4.6 and the fact that m_λ and h_λ are dual with respect to the Hall scalar product, we get

$$c_\lambda(t) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \sum_{\kappa \in \mathcal{PC}_{\delta_\sigma(\lambda)}} t^{\mathsf{des}(\kappa)}.$$

We construct a bijective correspondence between colorings $\kappa \in \mathcal{PC}_{\delta_\sigma(\lambda)}$ and P-arrays A of shape $\delta_\sigma(\lambda)$: fill the i-th row from the bottom from left to right with the vertices colored i in κ, in increasing order in P.

Under this correspondence, we have $\mathsf{des}(\kappa) = \mathsf{inv}_G(A)$.

So if $\mathcal{A}_{P,\lambda}$ is the set of pairs (A, σ) with A a P-array of shape $\delta_\sigma(\lambda)$, we have

$$c_\lambda(t) = \sum_{(A,\sigma)\in\mathcal{A}_{P,\lambda}} \operatorname{sgn}(\sigma) t^{\mathsf{inv}_G(A)}.$$

To get to 4.1, we define

$$\mathcal{B}_{P,\lambda} = \{(A,\sigma) \in \mathcal{A}_{P,\lambda} \mid A \notin \mathcal{T}_{P,\lambda}\}.$$

Notice that for any non-identity σ, $\delta_\sigma(\lambda)$ is not a partition (indeed if $i < j$ with $\sigma(i) > \sigma(j)$ then $\delta_\sigma(\lambda)_i < \delta_\sigma(\lambda)_j$). So all arrays in $\mathcal{A}_{\mathcal{P},\lambda}$ that are tableaux in $\mathcal{T}_{P,\lambda}$ are of shape $\delta_{\mathrm{Id}}(\lambda) = \lambda$.

Thus, to show 4.1, it suffices to define an involution $\psi : \mathcal{B}_{P,\lambda}$ sending (A, σ) to (A', σ') such that $\mathsf{inv}_G(A) = \mathsf{inv}_G(A')$ and $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\sigma')$.

For a P-array A, denote by $a_{i,j}$ the entry in the i-th row from the bottom and j-th column from the left. If $(A, \sigma) \in \mathcal{B}_{P,\lambda}$ it follows that there is at least one $a_{i,j}$ such that either $a_{i-1,j} >_p a_{i,j}$ or $a_{i-1,j} = 0$. Call such *bad entries.*

Take c minimal such that there exists a bad $a_{i,c}$ and r maximal such that $a_{r+1,c}$ is bad. For any $A \in \mathcal{A}_{\mathcal{P},\lambda}$ define

- I_r = entries in row r, weakly to the left of c
- O_r = entries in row r, strictly to the right of c, such that its connected component in G has an odd number of vertices
- E_r = entries in row r, strictly to the right of c, such that its connected component in G has an even number of vertices
- I_{r+1} = entries in row $r + 1$, strictly to the left of c
- O_{r+1} = entries in row $r + 1$, weakly to the right of c, such that its connected component in G has an odd number of vertices
- E_{r+1} = entries in row $r + 1$, weakly to the right of c, such that its connected component in G has an even number of vertices

So for $i \in \{r, r + 1\}$ the entries in the i-th row are $I_i \cup O_i \cup E_i$. Define A' to be the unique tableau whose entries in the r-th and $r + 1$-th row are $I_r \cup O_{r+1} \cup E_r$ and $I_{r+1} \cup O_r \cup E_{r+1}$, respectively. There are a few things to show.

1. The resulting A' is a tableau. We have to show that $I_r \cup O_{r+1} \cup E_r$ and $I_{r+1} \cup O_r \cup E_{r+1}$ form chains in P. For $i \in \{r, r + 1\}$, $x \in I_i$ and $y \in E_i$ implies that $x <_P y$, as A is a P-array. Furthermore, if $x \in O_i$ and $y \in E_i$ then x and y are comparable in P since they live in different connected components of G. We conclude by showing the following two statements:
   - *if $x \in I_r$ and $y \in O_{r+1}$ then $x <_p y$.* Since A is a P-array, we have
     $$a_{r+1,c-1} <_p a_{r+1,c} <_P a_{r+1,c+1}$$
     (if $c > 1$ and $a_{r+1,c+1} \neq 0$). By the minimality of c, we have $a_{r,c-1} \not>_p a_{r+1,c-1}$. Since P is (3+1)-free we must have $a_{c-1,r} <_p a_{r+1,c+1}$. Since $x \leq_p a_{c-1,r} <_P a_{r+1,c+1} \leq_P y$, the statement follows.
   - *if $x \in I_{r+1}$ and $y \in O_r$ then $x <_p y$.* Since $a_{r+1,c}$ is bad, if $a_{c,r} \neq 0$, we must have
     $$x \leq_P a_{r+1,c} <_P a_{r,c} \leq y.$$

2. The map ψ is an involution. We claim that $a'_{r+1,c} = a_{r+1,c}$ is bad. We have that $a'_{r,c} \in E_r \cup O_{r+1} \cup \{0\}$. If $a'_{r,c} \in O_{r+1}$ then $a'_{r,c} >_P a'_{r+1,c}$. If $a'_{r,c} \in E_r$ then $a'_{r,c} \geq_P a_{r,c} >_P a_{r+1,c} = a'_{r+1,c}$. So $a'_{r+1,c}$ is bad and satisfies the same minimal column and maximal row condition as before. Since $I_i(A') = I_i(A)$, $E_i(A') = E_i(A)$ for $i \in \{r, r+1\}$ and $O_r(A') = O_{r+1}(A)$, $O_{r+1}(A') = O_r(A)$, we must have $(A')' = A$.

3. If A is of shape $\delta_\sigma(\lambda)$ then A' is of shape $\delta_{\sigma'}(\lambda)$ with $\sigma' = \sigma\tau$, where τ is the transposition of r and $r+1$. Let $H_r(G)$ be the sub-graph of G containing only the vertices colored r or $r+1$. By Lemma 4.11, the connected components of $H_r(G)$ are chains with increasing vertex labels. Thus, each connected component of even size has half its vertices in row r and the other half in row $r+1$. So if we set ρ_i to be the number of nonzero entries in row i of A, we have

$$\begin{aligned}
\rho_r &= c - 1 + \delta_\sigma(\lambda)_{r+1} - c \\
&= \delta_\sigma(\lambda)_{r+1} - 1 \\
&= \delta_\sigma(\lambda)_{\tau(r)} + \tau(r) - r \\
\rho_{r+1} &= c + \delta_\sigma(\lambda)_r - (c-1) \\
&= \delta_\sigma(\lambda)_r + 1 \\
&= \delta_\sigma(\lambda)_{\tau(r+1)} + \tau(r+1) - (r+1)
\end{aligned}$$

4. We have $\mathsf{inv}_G(A) = \mathsf{inv}_G(A')$. The involution re-colors odd chains in $H_r(G)$ by inverting the r and $r+1$ colors. This operation conserves the inv_G.

Example 4.17. Take again the poset $P_{3,2}$ (see Example 4.15 and Figure 4.3). Take $\lambda = (1,1,1)$, then

σ	$\delta_\sigma(\lambda)$
Id	$(1,1,1)$
132	$(1,0,2)$
213	$(0,2,1)$
231	$(0,0,3)$
312	$(-1,2,2)$
321	$(-1,1,3)$.

It follows that $A_{P,\lambda}$ is the set of tableau depicted in Figure 4.4.

4.2 Colorings and orientations

Let us recall some definitions regarding colorings and orientations.

Definition 1.5. An *orientation* of a graph is a function $\rho\colon E \to V$ such that $\rho(e) \in e$, that is, a choice of a direction for each edge of the graph. We write $u \to v$ if $\{u,v\} \in E$ and $\rho(\{u,v\}) = v$, that is, if $\{u,v\}$ is an edge of the graph directed towards v.

Definition 1.6. An orientation ρ of a graph is *acyclic* if there is no subset $\{e_1, e_2, \ldots, e_k\} \subseteq E$ such that $e_{i+1} = \{\rho(e_i), \rho(e_{i+1})\}$ for all i (with $e_{k+1} := e_1$), that is, if there is no directed loop.

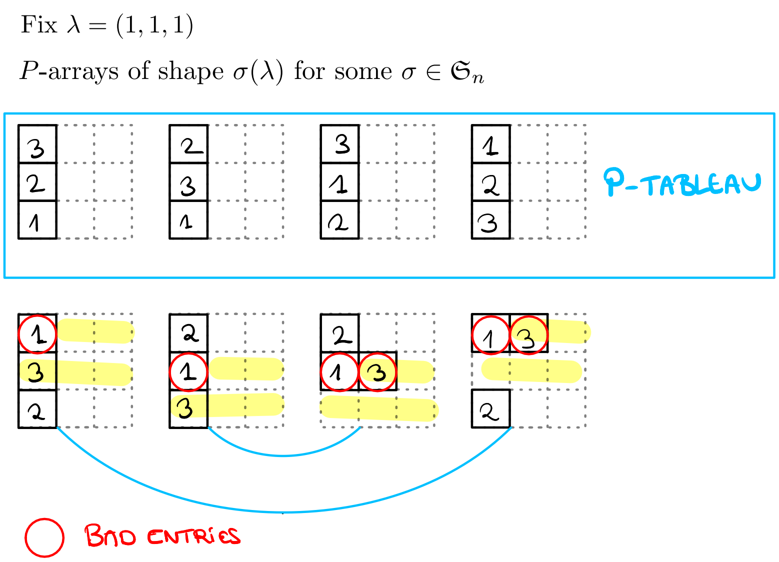


Figure 4.4: The ψ-involution on P-arrays

Definition 1.7. A coloring κ is *(strictly) compatible* with an acyclic orientation ρ if $u \to v \implies \kappa(u) \le \kappa(v)$ (resp. $\kappa(u) < \kappa(v)$).

Let is fix the following notation.

Notation 4.18. We denote by $\mathcal{AO}(G)$ the set of acyclic orientations of G.

Notation 4.19. For $\rho \in \mathcal{AO}(G)$, $\mathcal{PC}(\rho)$ is the set of *strictly* compatible colorings.

In Chapter 1, we observed that each proper coloring is compatible with exactly one acyclic orientation of G. This implies that

$$\mathcal{X}_G(X) = \sum_{\rho \in \mathcal{AO}(G)} \sum_{\kappa \in \mathcal{PC}(\rho)} x^\kappa. \tag{4.2}$$

Let us generalise this to the quasi-symmetric context.

Definition 4.20. For ρ an orientation of G, set $\mathsf{asc}(\rho)$ to be the number of $i \to j$ with $i < j$.

It is easy to see that for all $\kappa \in \mathcal{PC}(\rho)$, we have $\mathsf{asc}(\kappa) = \mathsf{asc}(\rho)$. Thus we can deduce

$$\mathcal{X}_G(X,t) = \sum_{\rho \in \mathcal{AO}(G)} t^{\mathsf{asc}(\rho)} \sum_{\kappa \in \mathcal{PC}(\rho)} x^\kappa. \tag{4.3}$$

In Chapter 1 we mentioned the following result without proof.

Theorem 1.17 ([Sta95, Theorem 3.3])**.** Let G be a graph with $\#V = n$. If $\mathcal{X}_G(X) = \sum_{\lambda \vdash n} c_\lambda e_\lambda$, then the number of acyclic orientations of G with exactly k sinks is $\sum_{\lambda \vdash n, \ell(\lambda)=k} c_\lambda$.

In this section we give a proof of the generalisation of this statement.

Theorem 4.21. Let $G = ([n], E)$. If $\mathcal{X}_G(X,t) = \sum_{\lambda \vdash n} c_\lambda(t) e_\lambda$ then

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=j}} c_\lambda(t) = \sum_{\rho \in \mathcal{AO}(G,j)} t^{\mathsf{asc}(\rho)},$$

where $\mathcal{AO}(G,j)$ is the set of acyclic orientations of G with j sinks.

We will need the following result from Chapter 3.

Proposition 3.16. Let $\varphi \colon \mathsf{QSym} \to \mathbb{Q}[q]$ defined on the fundamental basis as $\varphi(F_{n,S}) = q(q-1)^i$ if $S = \{i+1, i+2, \ldots, n-1\}$ and 0 otherwise. Then, if P is a poset with n elements, $\varphi(\mathcal{X}_P(X)) = q^m$, where m is the number of minimal elements of P.

We will apply this result to a certain poset induced by an orientation.

Definition 4.22. If $\rho \in \mathcal{AO}(G)$ for some graph $G = ([n], E)$, then it is easy to see that the transitive closure of relations $\{i < j \mid i \to j\}$ forms a poset on $[n]$. Denote this poset by $\overline{\rho}$.

It is clear from this definition of the poset $\overline{\rho}$, and the definition of $\mathcal{X}_P(X)$, the quasisymmetric function associated to any poset P (Definition 3.13) that

$$\sum_{\kappa \in \mathcal{AO}(\rho)} x^{\kappa} = \mathcal{X}_{\overline{\rho}}(X) \tag{4.4}$$

Proof of Theorem 4.21. By 4.3 and 4.4, we have

$$\mathcal{X}_G(X,t) = \sum_{\rho \in \mathcal{AO}(G)} t^{\mathsf{asc}(\rho)} \mathcal{X}_{\overline{\rho}}(X).$$

Applying ϕ from Proposition 3.16 we get

$$\phi(\mathcal{X}_G(X,t)) = \sum_{\rho \in \mathcal{AO}(G)} t^{\mathsf{asc}(\rho)} q^{\#\text{minimal elments of } \overline{\rho}} = \sum_{\rho \in \mathcal{AO}(G)} t^{\mathsf{asc}(\rho)} q^{\#\text{sinks of } \rho}.$$

Where the second equality comes from the fact that the definition of $\overline{\rho}$ implies

$$\#\text{ minimal elements of } \overline{\rho} = \#\text{ sinks of } \rho.$$

Now for any $\lambda \vdash n$ consider P_λ to be the disjoint union of $\ell(\lambda)$ chains of lengths $\lambda_1, \lambda_2, \ldots$. Clearly, $\mathcal{X}_{P_\lambda}(X) = e_\lambda$, so

$$\phi(\mathcal{X}_G(X,t)) = \sum_{\lambda \vdash n} c_\lambda(t) \phi(e_\lambda) = \sum_{\lambda \vdash n} c_\lambda(t) \phi(\mathcal{X}_{P_\lambda}) = \sum_{\lambda \vdash n} c_\lambda(t) q^{\ell(\lambda)}.$$

Comparing the q-coefficients of the last two equations yields the result.

4.3 Gessel expansion

Definition 4.23. Given a graph $G = ([n], E)$ and a permutation $\sigma \in \mathfrak{S}_n$, a G-inversion is a pair $\{i, j\} \in E$ with $i < j$ and $\sigma(i) > \sigma(j)$.

Definition 4.24. Given poset P on $[n]$ and a permutation $\sigma \in \mathfrak{S}_n$, the P-*descent set* is defined as

$$\mathsf{Des}_P(\sigma) := \{i \in [n-1] \mid \sigma(i) >_p \sigma(i+1)\}.$$

For $\sigma \in \mathfrak{S}_n$ set $S(\sigma) := \{i \in [n-1] \mid n - i \in \mathsf{Des}_P(\sigma)\}$

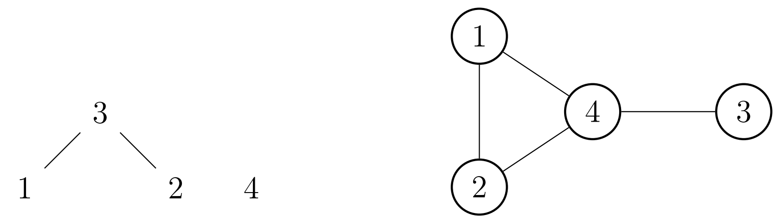


Figure 4.5: Hasse diagram of a poset P (left) and its incomparability graph G (right).

Theorem 4.25. Let G be the incomparability graph of a poset on $[n]$, then

$$\mathcal{X}_G(X,t) \coloneqq \sum_{s\in\mathfrak{S}_n} t^{\mathsf{inv}_G(s)} F_{S(s)}$$

To prove this theorem, we will use the following terminology, which seems redundant at first but is key to the right understanding.

Definition 4.26. Let $G = (V = [n], E)$ be a graph on n vertices. A *labelling* and a *sequencing* are both permutations of n, but we think of a labelling as a map $V \to [n]$ and of a sequencing as a map $[n] \to V$.

Apply the same definitions to a poset P on $[n]$: a sequencing $[n] \to P$ and a labelling $P \to [n]$.

We will represent a sequencing $s : [n] \to V$ as a linear order on the vertices (or elements of the poset), and a labelling $w : V \to [n]$ as a label attached to the vertex (or element of the poset).

We will reformulate the following result from Chapter 3 in terms of labellings and sequencings.

Proposition 3.15. Let ψ be a linear extension of P^*, i.e. an order-reversing bijection $\psi\colon P \to [n]$. For α linear extension of P, define $\sigma_\psi(\alpha) = (\psi(\alpha^{-1}(i)))_{1\le i\le n} \in S_n$. We have

$$\mathcal{X}_P(X) = \sum_{\substack{\alpha \text{ linear} \\ \text{extension}}} F_{\mathsf{Des}(\sigma_\psi(\alpha))}.$$

In Definition 3.14, a linear extension is defined as an order-preserving bijection $P \to [n]$, so we can think of $s \coloneqq \alpha^{-1} : [n] \to P$ as an order-preserving sequencing and $\psi : P \to [n]$ as an order reversing labelling of P.

$$\mathcal{X}_P(X) = \sum_{\substack{s \text{ order preserving} \\ \text{sequencing of } P}} F_{\mathsf{Des}(\psi\circ s)} \tag{4.5}$$

Example 4.27. Consider the poset P on [4] depicted in Figure 4.5. In Figure 4.6, we represent the sequencing 1423 and the labelling 4213 of its incomparability graph.

Proof. We start from 4.3

$$\mathcal{X}_G(X,t) = \sum_{\rho\in\mathcal{AO}(G)} t^{\mathsf{asc}(\rho)} \sum_{\kappa\in\mathcal{PC}(\rho)} x^\kappa.$$

Recall that any orientation ρ induces a poset $\overline{\rho}$ on the vertices of G by transitive closure (Definition 4.22).

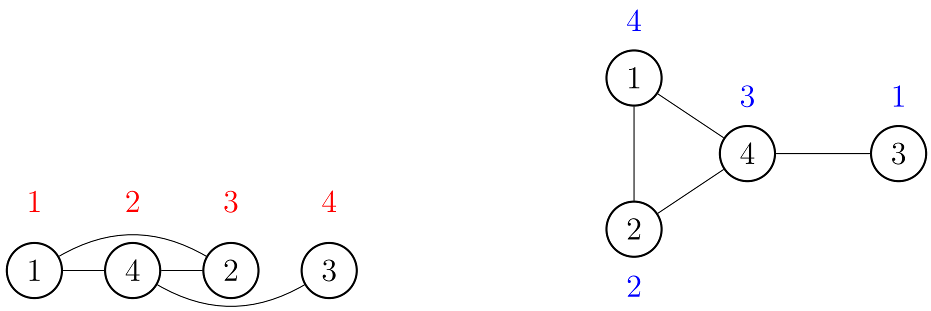


Figure 4.6: A sequencing (left) and a labelling (right) of G.

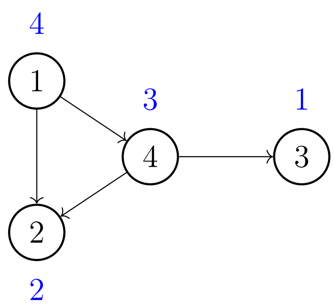


Figure 4.7: An acyclic orientation of G and a decreasing labelling.

Fix an acyclic orientation $\rho \in \mathcal{AO}(G)$ and take w_ρ an order-reversing labelling with respect to $\overline{\rho}$, that is $i \to j$ implies $w_\rho(i) > w_\rho(j)$. For example, in Figure 4.7 we depicted an orientation ρ and a decreasing labelling w_ρ.

Set $\mathcal{S}_\rho$ to be the set of sequencings $s : [n] \to V$ that are order-preserving with respect to $\overline{\rho}$, that is $s(i) \to s(j)$ implies $i < j$. In other words if s is represented as a linear order on the vertices from left to right, all oriented edges of G point to the right. For example if ρ is the orientation depicted in 4.7 (with its decreasing labelling), then Figure4.8 depicts the sequencings in $\mathcal{S}_\rho = \{1423, 1432\}$ (and the decreasing labelling in blue below the graph).

Taking $P = \overline{\rho}$ in 4.5

$$\mathcal{X}_{\overline{\rho}}(X) = \sum_{\kappa \in \mathcal{PC}(\rho)} x^\kappa = \sum_{s \in S_\rho} F_{\mathsf{Des}(w_\rho \circ s)} \tag{4.6}$$

In our example of Figure 4.8, we see that $\{w_\rho \circ s \mid s \in S_\rho\} = \{4321, 4312\}$.

Let us make a few observations regarding $\mathcal{S}_\rho$.

- We have that $\{\mathcal{S}_\rho \mid \rho \in \mathcal{AO}(G)\}$ partitions the set of sequencings (permutations). For any sequencing s, denote by $\rho(s)$ the unique acyclic orientation such that $s \in \mathcal{S}_\rho$.
- We have $\mathsf{asc}(\rho(s)) = \mathsf{inv}_G(s^{\mathsf{rev}})$.

Figure 4.8: The sequencings s in S_ρ for ρ of Figure 4.7 and its decreasing labelling $w_{\rho(s)}$, yielding permutations $\sigma = w_{\rho(s)} \circ s$

Combining these observations with 4.3 and 4.6, we get

$$\begin{aligned}\mathcal{X}_G(X,t) &= \sum_{\rho\in\mathcal{AO}(G)} t^{\mathsf{asc}(\rho)} \sum_{s\in\mathcal{S}_\rho} F_{\mathsf{Des}(w_\rho\circ s)} = \sum_{s\in\{\mathcal{S}_\rho|\rho\in\mathcal{AO}(G)\}} t^{\mathsf{inv}_G(s^{\mathsf{rev}})} F_{w_{\rho(s)}\circ s} \\ &= \sum_{s \text{ sequencing of } G} t^{\mathsf{inv}_G(s^{\mathsf{rev}})} F_{w_{\rho(s)}\circ s}. \end{aligned} \tag{4.7}$$

Until now, w_ρ was just any $\overline{\rho}$-reversing labelling. Let us now fix a canonical choice $\tilde{w}_\rho$ for such a labelling.

(1) Set $i = 1$

(2) Among the maxima of $\overline{\rho}$ (i.e. the sinks in G), label with i the element that is largest in P (since they are incomparable in $\overline{o}$ they must be comparable in P).

(3) Remove the newly labelled element from the poset $\overline{\rho}$, increase i by one and go back to (2).

Claim. For all x, y incomparable in $\overline{\rho}$, $x <_P y \Rightarrow \tilde{w}_\rho(x) > w_\rho(y)$.

Suppose by contradiction that there are x, y, incomparable in $\overline{\rho}$ with $x <_P y$ and $\tilde{w}_\rho(x) < \tilde{w}_\rho(y)$.

Consider the moment in time at which x gets its label $\tilde{w}_\rho(x)$ in the definition process described above. Since $\tilde{w}_\rho(x) < \tilde{w}_\rho(y)$ at this point y is unlabelled.

Since $x <_P y$, we have that y is not maximal in the current poset $\overline{\rho}$. Thus there must be at least one unlabelled z such that $z <_P x$ and $y <_{\overline{\rho}} z$. Pick z to be $\overline{\rho}$ minimal.

Take a saturated chain $y <_{\overline{\rho}} y_1 <_{\overline{\rho}} \cdots <_{\overline{\rho}} y_k = z$. The saturation implies that each element of the chain is connected to the next by an edge in G.

The y_i's are unlabelled since z is. So $x \not<_{\overline{\rho}} y_i$ for all i.

Since x and y are incomparable in $\overline{\rho}$ we may not have $y_i <_{\overline{\rho}} x$ for any i. Combined with the previous observation we deduce that y_i and x are incomparable in $\overline{\rho}$.

Not comparable in $\overline{\rho}$ means no edge in G and so comparable in P. So y_i and x are comparable in P for all i.

The minimality of z implies that $y_i \not<_P x$ for $i < k$. So $x \leq_P y_{k-1}$. Thus $z <_P x$ implies $z <_P y_{k-1}$, which contradicts the fact that $\{z, y_{k-1}\}$ forms an edge in G. Thus we have established the claim.

Let us now show that for any sequencing s of G

$$\mathsf{Des}(\tilde{w}_{\rho(s)} \circ s) = [n-1] \setminus \mathsf{Des}_P(s) \tag{4.8}$$

We distinguish two cases.

- *The vertices $s(i)$ and $s(i+1)$ are not comparable in P.* That means $i \notin \mathsf{Des}_P(s)$. Furthermore the incomparability in P means that $\{s(i), s(i+1)\}$ must be an edge in G. Since $\tilde{w}_{\rho(s)}$ is decreasing on the orientation induced by s $(s(i) \to s(i+1))$, we must have $\tilde{w}_{\rho(s)}(s(i)) > \tilde{w}_{\rho(s)}(s(i+1))$ and so $i \in \mathsf{Des}(\tilde{w}_{\rho(s)} \circ s)$.

- *The vertices $s(i)$ and $s(i+1)$ are comparable in P.* Thus, in G there is no edge between the vertices $s(i)$ and $s(i+1)$. Since i and $i+1$ are consecutive this means that $s(i)$ and $s(i+1)$ are incomparable in $\rho(s)$. Thus by the claim, $i \in \mathsf{Des}_P(s) \Leftrightarrow i \in \mathsf{Des}(\tilde{w}_{\rho(s)} \circ s)$.

So using 4.7

$$\mathcal{X}_G(X,t) = \sum_{s \text{ sequencing of } G} t^{\mathsf{inv}_G(s^{\mathsf{rev}})} F_{[n-1]\setminus \mathsf{Des}_P(s)}.$$

We have $i \in \mathsf{Des}_P(s) \Leftrightarrow n-i \in \mathsf{Des}_{P*}(s^{\mathsf{rev}})$, where P^* is the dual of P. Thus, since the set of sequences of G are just permutations of n, we may sum on s^{rev} and the result follows from the fact that the incomparability graph of P and P^* coincide.

Example 4.28. If P and G are as in Figure 4.5

$$\begin{aligned}\mathcal{X}_G(X,t) =& (t^4+3t^3+4t^2+3t+1)F_{1,1,1,1} + (t^2+2t+1)F_{1,1,2} \\ &+ (t^4+t^3+t+1)F_{1,2,1} + (t^4+2t^3+t^2)F_{2,1,1}.\end{aligned}$$

The motivated (or bored) reader can check that these are the 24 expected terms.

Chapter 5

Plethysm

Plethystic notation is an extremely powerful tool that has been introduced to write symmetric function identities in a more compact way. It also provides a convenient way to deal with composition of Schur functors, it can be used to endow Λ with a Hopf algebra structure and a λ-ring structure, and to derive combinatorial identities with much shorter proofs; it also helps understanding the combinatorial reciprocity we're seeing at the *COmbinatorics Seminar for Everyone*.

5.1 The combinatorial approach

In this section we will give an informal definition of plethystic substitution, in order to get accustomed with the notation and see the pragmatic way to actually compute these things. A nice combinatorial presentation on plethystic calculus is given in [LR11]. The approach we will see, while not exactly formal, is intuitive; we'll see a formal construction later with λ-rings.

The combinatorial approach to plethysm consists of treating alphabet as sums: for example, we identify the alphabet $X = (x_1, x_2, \dots)$ with the symmetric function $e_1 = x_1 + x_2 + \dots$; from now on, we will write $X = x_1 + x_2 + \dots$ even when we refer to the alphabet. If f is a symmetric function and X is an alphabet, we call $f[X]$ the *plethystic evaluation* of f in X.

In this notation, the sum of functions corresponds to concatenation of alphabets, so if f is a symmetric function and $X = x_1 + x_2 + \dots$, $Y = y_1 + y_2 + \dots$ are two alphabets, then $f[X+Y] = f(x_1, x_2, \dots, y_1, y_2, \dots)$. This means that a sum of monomials is itself an alphabet, so we can plug in one into the other: for example, we have that, since $h_2 = x_1^2 + x_1x_2 + x_2^2 + \dots$, then

$$e_2[h_2] = (x_1^2)(x_1x_2) + (x_1^2)(x_2^2) + (x_1x_2)(x_2^2) + \dots.$$

Let's try to give a slightly more formal definition.

Definition 5.1. If $F(t_1, t_2, \dots) \in A((t_1, t_2, \dots))$ is any formal Laurent series on any ring A, we define

$$p_n[F] \coloneqq F(t_1^n, f_2^n, \dots),$$

and if $f = \sum_\lambda a_\lambda p_\lambda \in \Lambda$, then we define

$$f[F] \coloneqq \sum_\lambda a_\lambda \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}[F].$$

Equivalently, $f[F]$ is the image of the algebra homomorphism from Λ to $A((t_1, t_2, \dots))$ mapping p_n to the formal Laurent series obtained from F by rising every variable to the n-th power.

Note that plethystic substitution *does not commute with evaluation of variables.* Indeed,

$$p_n[x]|_{x=2} = x^n|_{x=2} = 2^n,$$

while $p_n[2] = 2$ because 2 is not a variable.

The idea behind this is that, if $f \in \Lambda$, then $f[x_1 + x_2 + \dots] = f(x_1, x_2, \dots)$. More generally, if F has an expression as sum of monomials all with coefficient 1, then $f[F]$ is the expression obtained by replacing the x_i's with such monomials.

The plethystic evaluation has several nice properties.

- If $g \in \Lambda \subseteq \mathbb{K}((x_1, x_2, \dots))$, then $f[g] \in \Lambda$. This operation, called *plethysm*, is associative.
- If $f \in \Lambda^{(d)}$, then $f[uX] = u^d f[X]$ for any indeterminate u, and $f[-X] = (-1)^d \omega f[X]$. Notice that, again, evaluating the indeterminates does not commute with the plethystic evaluation.
- The coproduct $\Delta(f[X]) = f[X + Y]$ and the antipodal map $S(f[X]) = f[-X]$ define a Hopf algebra structure on Λ.
- Let ϵ be the automorphism defined by $f[\epsilon X] \coloneqq \omega f[-X]$. It corresponds to the substitution $x_i \mapsto -x_i$ (which is not the same as $X \mapsto -X$).

Since the sum of two alphabets can be seen as the concatenation, we can easily derive the following summation formulae.

Proposition 5.2. For $n \in \mathbb{N}$, the following summation formulae hold.

$$e_n[X + Y] = \sum_{i=0}^{n} e_i[X] e_{n-i}[Y] \quad \text{and} \quad h_n[X + Y] = \sum_{i=0}^{n} h_i[X] h_{n-i}[Y].$$

A detailed proof of this statement can be found in [LR11]. Now recalling that, if $f \in \Lambda^{(n)}$, then $f[-X] = (-1)^n \omega f[X]$, we immediately get the following corollary.

Corollary 5.3. For $n \in \mathbb{N}$, the following subtraction formula holds.

$$e_n[X - Y] = \sum_{i=0}^{n} (-1)^{n-i} e_i[X] h_{n-i}[Y]$$

To deal with the products, we need the *Cauchy identity.*

Theorem 5.4 (Cauchy identity)**.** Let $\{u_\lambda \mid \lambda \vdash n, n \in \mathbb{N}\}$, $\{v_\lambda \mid \lambda \vdash n, n \in \mathbb{N}\}$ be a pair of dual bases of Λ with respect to the Hall scalar product. Then for $n \in \mathbb{N}$,

$$h_n[XY] = \sum_{\lambda \vdash n} u_\lambda[X] v_\lambda[Y].$$

Proof. First of all, notice that the following are equivalent:

1. There exists a pair of dual bases $\{u_\lambda\}$, $\{v_\lambda\}$ such that $h_n[XY] = \sum_{\lambda \vdash n} u_\lambda[X]v_\lambda[Y]$;
2. For every pair of dual bases $\{u_\lambda\}$, $\{v_\lambda\}$ the identity $h_n[XY] = \sum_{\lambda \vdash n} u_\lambda[X]v_\lambda[Y]$ holds;
3. For every $f \in \Lambda$, $\langle h_n[XY], f[X]\rangle = f[Y]$.

It is immediate that (3.) $\implies$ (2.) $\implies$ (1.) $\implies$ (3.), hence the statements are all equivalent. To prove the statement is therefore sufficient to show (1.) for a pair of dual bases of our choice; we will do that for $\{p_\lambda\}$ and $\{\frac{p_\lambda}{z_\lambda}\}$.

In particular, we will show that

$$\sum_{n \in \mathbb{N}} h_n[XY] = \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j} = \sum_\lambda \frac{1}{z_\lambda} p_\lambda[X] p_\lambda[Y]$$

and the statement will follow by isolating the part in degree n (because Λ is graded).

The first equality is trivial as $XY = \sum x_i y_j$ and the infinite product is precisely the generating function for the complete homogeneous symmetric functions. The second equality requires a little more work. We have

$$\begin{aligned}
\prod_{i,j} \frac{1}{1 - x_i y_j} &= \prod_{i,j} \exp(-\log(1 - x_i y_j)) \\
&= \prod_{i,j} \exp\left(\sum_k \frac{(x_i y_j)^k}{k}\right) = \exp\left(\sum_{i,j,k} \frac{(x_i y_j)^k}{k}\right) \\
&= \exp\left(\sum_k \frac{p_k[X] p_k[Y]}{k}\right) = \sum_n \frac{1}{n!} \left(\sum_k \frac{p_k[X] p_k[Y]}{k}\right)^n \\
&= \sum_n \frac{1}{n!} \sum_{\sum \alpha_k = n} \binom{n}{\alpha_1, \ldots, \alpha_\ell} \prod_{k=0}^{\ell} \left(\frac{p_k[X] p_k[Y]}{k}\right)^{\alpha_k} \\
&= \sum_\lambda \frac{1}{z_\lambda} p_\lambda[X] p_\lambda[Y]
\end{aligned}$$

where in the last step we collect the compositions with the same parts sizes. This proves the theorem.

5.2 Representations of $GL_n(\mathbb{C})$

For each partition λ of length $\ell(\lambda) \leq n$ there exists an irreducible $GL_n(\mathbb{C})$-representation S^λ that admits a basis

$$\mathcal{E}_\lambda = \{e_T \mid T \in \mathsf{SSYT}(\lambda, [n])\}$$

satisfying $\mathsf{diag}(x_1, \ldots, x_n) e_T = x^T e_T$.

The character of S^λ is given by the trace of the matrix representing this action, so

$$\chi_\lambda(\mathsf{diag}(x_1, \ldots, x_n)) = \sum_{T \in \mathsf{SSYT}(\lambda, [n])} x^T = s_\lambda(x_1, \ldots, x_n).$$

Example 5.5. Let $n = 2$ and $\lambda = 2$, and let $\rho_{(2)} \colon GL_2(\mathbb{C}) \to GL_3(\mathbb{C})$ be our representation, where $3 = \#\mathsf{SSYT}((2), [2])$. We have

$$\rho_{(2)}\left(\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 & 0 & 0 \\ 0 & x_1x_2 & 0 \\ 0 & 0 & x_2^2 \end{bmatrix}$$

whose trace is $s_2(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$.

For $n = 3$ and $\lambda = (2, 1)$, we have $\mathsf{diag}(y_1, y_2, y_3) \mapsto \mathsf{diag}(y^T \mid T \in \mathsf{SSYT}((2, 1), [3]))$.

We can compose these characters! The result is given by the substitution $y_1 = x_1^2, y_2 = x_1x_2, y_3 = x_2^2$, so

$$\rho_{(2,1)} \circ \rho_{(2)}(\mathsf{diag}(x_1, x_2)) = s_2(x_1^2, x_1x_2, x_2^2) = (s_{321} + s_{42} + s_{51})(x_1, x_2).$$

This operation on characters extends to Λ, and behaves as plethysm: we are substituting the variables of one symmetric function with the monomials appearing in another symmetric functions. So, we can deduce the following.

Proposition 5.6. Let $GL_n(\mathbb{C}) \xrightarrow{\rho} GL_m(\mathbb{C})$ and $GL_m(\mathbb{C}) \xrightarrow{\tau} GL_\ell(\mathbb{C})$ be two representations such that $\tau \circ \rho$ is defined. Then $\chi_{\tau\circ\rho} = \mathsf{Frob}(\chi_\tau)[\mathsf{Frob}(\chi_\rho)]$, where the square brackets denote the plethystic substitution, and $\mathsf{Frob}(S^\lambda) := s_\lambda$.

This is, very roughly, the representation-theoretical motivation for plethysm.

5.3 λ-rings

The possibly cleanest way to make plethystic substitution formal is via λ-rings, which are a "decategorification" of external products. Essentially, we want a ring structure mimicking the way the external product behaves under

- direct sum:
$$\Lambda^n(V \oplus W) = \bigoplus_{k=0}^{n} \Lambda^k(V) \otimes \Lambda^{n-k}(W);$$
- tensor product:
$$\Lambda^n(V \otimes W) = P_n(\Lambda^1(V), \ldots, \Lambda^n(V), \Lambda^1(W), \ldots, \Lambda^n(W));$$
- iteration:
$$\Lambda^n(\Lambda^m(V)) = P_{n,m}(\Lambda^1(V), \ldots, \Lambda^{mn}(V)).$$

These identities hold for some universal polynomials with integer coefficients P_n and $P_{n,m}$. We can construct them as follows.

Proposition 5.7. The polynomials P_n, $P_{n,m}$ satisfy

$$\prod_{I \subseteq \binom{[mn]}{m}} \left(1 + t \prod_{i \in I} x_i\right) = \sum_{n \in \mathbb{N}} P_{n,m}(e_1[X], \ldots, e_{mn}[X]) t^n$$

where $X = (x_1, \dots, x_{mn})$, and

$$\prod_{i,j=1}^{n}(1 + tx_iy_j) = \sum_{n\in\mathbb{N}} P_n(e_1[X], \dots, e_n[X], e_1[Y], \dots, e_n[Y])t^n$$

where $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$.

We can now give the definition of λ-ring.

Definition 5.8. A *λ-ring* is a ring R equipped with a family of operations $\lambda^n \colon R \to R$ such that:

- $\lambda^0(x) = 1$ for all $x \in R$;
- $\lambda^1(x) = x$ for all $x \in R$;
- $\lambda^n(1) = 0$ for all $n \geq 2$;
- $\lambda^n(x + y) = \sum_{k=0}^{n} \lambda^k(x)\lambda^{n-k}(y)$;
- $\lambda^n(xy) = P_n(\lambda^1(x), \dots, \lambda^n(x), \lambda^1(y), \dots, \lambda^n(y))$;
- $\lambda^n(\lambda^m(x)) = P_{n,m}(\lambda^1(x), \dots, \lambda^{mn}(x))$.

For example, $\mathbb{Z}$ is a λ-ring with the operation $\lambda^n(x) = \binom{x}{n}$ (and this is the only possible λ-ring structure on $\mathbb{Z}$), which can be derived by the ring generated by the finite dimensional vector spaces over a field $\mathbb{K}$ up to isomorphism by applying the dimension map, as $\dim(\lambda^n(\mathbb{K}^x)) = \binom{x}{n}$.
Indeed, Λ is a λ-ring with the operation $\lambda^n(z) = \binom{z}{n}$ if $z \in \mathbb{Q}$, and $\lambda^n(e_1) = e_n$. In fact, $\Lambda_\mathbb{Z}$ is the free λ-ring generated by one element, and $\Lambda = \Lambda_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$.
With these definitions, each element F of any λ-ring R determines a unique morphism $\phi_F \colon \Lambda \to R$ defined by $\phi_F(e_1) = F$, so for $R = \Lambda$ we get plethystic substitution via $f[g] \coloneqq \phi_g(f)$.

5.4 Combinatorial formulae for the monomial and the forgotten basis

Using the Cauchy identity, we can derive a combinatorial formula for any basis that is dual to a multiplicative basis, as follows.

Proposition 5.9 (Summation formula)**.** Let $\{u_\lambda\}_\lambda$ be a multiplicative basis of Λ, and let $\{v_\lambda\}_\lambda$ be its dual basis. Then

$$v_\lambda[X + Y] = \sum_{\mu\cup\nu=\lambda} v_\mu[X]v_\nu[Y].$$

Proof. By the Cauchy identity, for any alphabets X, Y, Z, we have

$$h_n[Z(X + Y)] = \sum_{\lambda\vdash n} u_\lambda[Z]v_\lambda[X + Y].$$

On the other hand, we also have

$$\begin{aligned}
h_n[Z(X+Y)] &= h_n[ZX+ZY] \\
&= \sum_{k=0}^{n} h_k[ZX]h_{n-k}[ZY] \\
&= \sum_{k=0}^{n} \left(\sum_{\mu \vdash k} u_\mu[Z]v_\mu[X] \right) \left(\sum_{\nu \vdash n-k} u_\mu[Z]v_\mu[Y] \right) \\
&= \sum_{\lambda \vdash n} u_\lambda[Z] \sum_{\mu \cup \nu = \lambda} v_\mu[X]v_\nu[Y].
\end{aligned}$$

Since $\{u_\lambda\}_\lambda$ is a basis, the thesis follows.

Iterating the summation formula, we get the following.

Proposition 5.10. If $\{v_\lambda\}_\lambda$ is dual to a multiplicative basis, then

$$v_\lambda[X] = \sum_{\lambda = \bigcup \mu^i} \prod v_{\mu^i}[x_i] = \sum_{\lambda = \bigcup \mu^i} \prod x_i^{|\mu^i|} v_{\mu^i}[1].$$

This means that, thanks to the summation formula, a basis that's dual to a multiplicative basis is uniquely determined by the evaluation at 1.

5.4.1 The monomial basis

Let's double check that the dual basis of the homogeneous are the monomial: let's call $\tilde{m}_\lambda$ the dual of h_λ, and let's check that $\tilde{m}_\lambda = m_\lambda$. We have

$$h_n[X] = \sum_{\lambda \vdash n} h_\lambda[X]\tilde{m}_\lambda[1]$$

so, since the homogeneous are a basis, we have that $\tilde{m}_\lambda[1] = 1$ if λ has (at most) one part, and $\tilde{m}_\lambda[1] = 0$ otherwise.

Using the summation formula, we get

$$\tilde{m}_\lambda[X] = \sum_{\bigcup(\alpha_i) = \lambda} x_1^{\alpha_1} x_2^{\alpha_2} \cdots = \sum_{\lambda(\alpha) = \lambda} \sum_{i_1 < i_2 < \ldots} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots$$

which is the usual monomial symmetric function.

5.4.2 The forgotten basis

The dual basis of the elementary basis is known as forgotten basis, and it is denoted by $\{f_\lambda\}_\lambda$. In order to find a combinatorial formula for that, we need to compute $f_\lambda[1]$. As before, we have

$$h_n[X] = \sum_{\lambda \vdash n} e_\lambda[X] f_\lambda[1]$$

so we need to find the e-expansion for $h_n[X]$.

Lemma 5.11.
$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \binom{n}{\lambda} e_\lambda.$$

Proof. We prove it via a sign-reversing involution. Recall that $e_n = s_{1^n}$; the right-hand side is given by (signed) sequences of semi-standard column tableaux of global size n, and the sign is given by n minus the number of tableaux.

Define a map ψ on the set of sequences of semi-standard column tableaux (of any size) as follows: given $T = (T_1, \dots, T_r)$ such a sequence, define $\psi(T)$ by the following process:

1. if $r = 0$, then $\psi(T) = T$;
2. if T_1 has at least two cells, $\psi(T) = (S_1, S_2, T_2, \dots, T_r)$, where S_1 is the bottom cell of T_1 and S_2 is the rest of T_1;
3. if T_1 has one cell, and its entry is strictly bigger than the entry in the top cell of T_2, then $\psi(T) = (T_1 \cup T_2, T_3, \dots, T_r)$;
4. otherwise we inductively define $\psi(T) = (T_1, \psi(T_2, \dots, T_r))$.

It is clear that ψ is a sign-reversing involution. Its fixed points are the sequences $(T_1, \dots, T_r)$ where every tableau has one cell, say T_i with entry a_i, and $a_1 \leq a_2 \leq \cdots \leq a_r$, which is the same as $s_r = h_r$. By restricting to sequences with n cells in total, the thesis follows.

Corollary 5.12. For $\lambda \vdash n$, we have
$$f_\lambda[1] = (-1)^{n-\ell(\lambda)} \binom{n}{\lambda}.$$

Proposition 5.13. Using the summation formula, we get
$$f_\lambda[X] = \sum_{\bigcup \mu^i = \lambda} \prod_i (-1)^{|\mu^i| - \ell(\mu^i)} \binom{|\mu^i|}{\mu^i} x_i^{|\mu^i|} = \sum_{\lambda(\alpha)=\lambda} (-1)^{n-\ell(\lambda)} \sum_{i_1 \leq i_2 \leq \dots} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots$$

Since $\omega(h_n) = e_n$, then $\omega(m_\lambda) = f_\lambda$, so once again we see that applying ω swaps strict inequalities and weak inequalities.

Chapter 6

LLT polynomials

In this chapter, we will study a family of symmetric functions known as LLT-polynomials (after the authors who first introduced them in 1997: Lascoux, Leclerc, Thibon [LLT97]). These symmetric functions can be seen as a q-deformation of products of (skew) schur functions.

6.1 Definition

Definition 6.1. Given the a cell $u = (i, j)$ in the Young diagram of a (skew) partition λ define the *content* of u to be $c(u) \coloneqq i - j$.

Example 6.2. For $\lambda = (4, 3, 2)$ we have labelled each cell with its content:

2	1		
1	0	-1	
0	-1	-2	-3

Definition 6.3. Given $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(k)})$ a tuple of skew partitions, define the set of *SSYT-tuples*

$$\mathsf{SSYT}(\boldsymbol{\nu}) \coloneqq \mathsf{SSYT}(\nu^{(1)}) \times \cdots \times \mathsf{SSYT}(\nu^{(k)}).$$

We represent these kind of tuples by drawing them, from **right to left** and aligning the boxes of the same content of each tableau. See Figure 6.1 for an example of a tuple of tableaux of shape $((2,2), (2,1,1) \setminus (1), (2,3))$.

For $\mathbf{T} = (T_1, \ldots, T_k) \in \mathsf{SSYT}(\boldsymbol{\nu})$ we define its monomial to be $x^{\mathbf{T}} \coloneqq x^{T_1} \cdots x^{T_k}$.

Definition 6.4. Given $\mathbf{T} = (T_1, \ldots, T_k) \in \mathsf{SSYT}(\boldsymbol{\nu})$ for some tuple of partitions $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(k)})$, a pair of cells (u, v) with $u \in \nu^{(i)}$ and $v \in \nu^{(j)}$ and $i < j$ is called an *inversion pair* if one of the following holds

- $c(u) = c(v)$ and $T_i(u) > T_j(v)$
- $c(u) + 1 = c(v)$ and $T_i(u) < T_j(v)$.

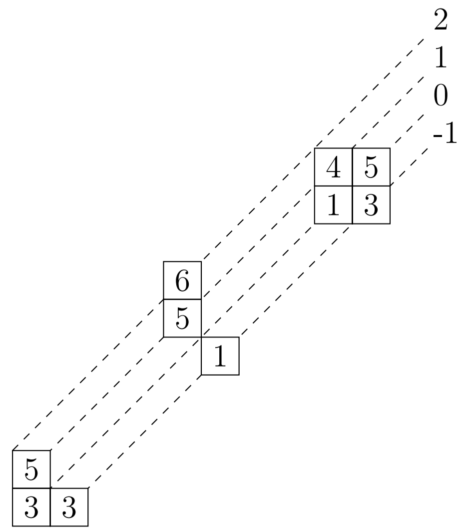


Figure 6.1: The tuple of skew tableau of shape $((2,2),(2,1,1)\setminus(1),(2,3))$

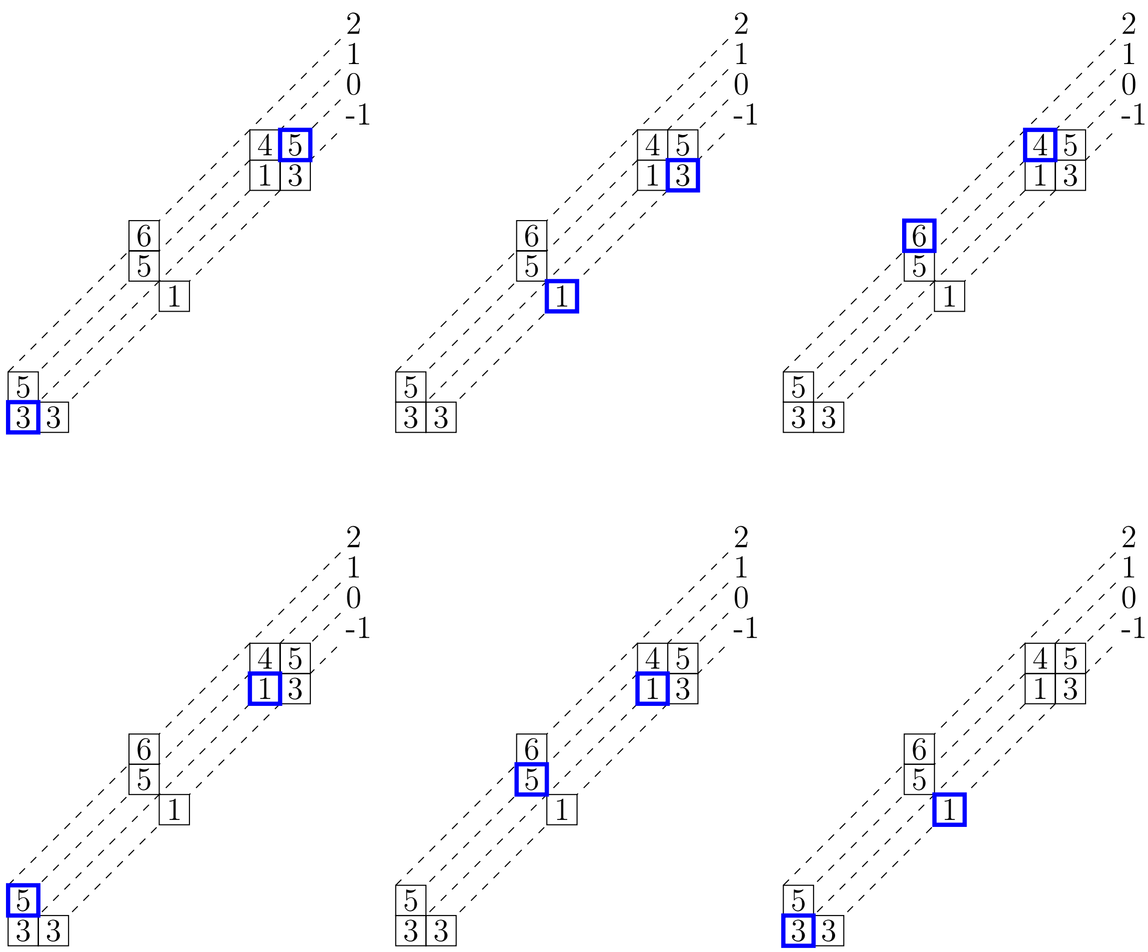


Figure 6.2: The inv of a tuple of SSYT

Define $\mathsf{inv}(\mathbf{T})$ to be its number of inversion pairs.

In Figure 6.2 we we highlight the inversion pairs of the tableau tuple of Figure 6.1. We conclude that its inv is equal to 5.

Definition 6.5. Given a tuple of (skew) diagrams $\boldsymbol{\nu}$, define the *LLT-polynomial* indexed by $\boldsymbol{\nu}$

$$LLT_{\boldsymbol{\nu}}[X] = \sum_{\mathbf{T} \in \mathsf{SSYT}(\boldsymbol{\nu})} q^{\mathsf{inv}(\mathbf{T})} x^{\mathbf{T}}.$$

6.2 Symmetry

The goal of this section is to prove the following result.

Theorem 6.6. For all tuples of diagrams ν, LLT_ν is a symmetric function.

We follow the proof in the appendix of [HHL05].

6.2.1 Superization

Consider the alphabet $\mathcal{A} = \{1, \overline{1}, 2, \overline{2}, 3, \overline{3}, \dots\}$ of *barred* (or *negative*) and *unbarred* (or *positive*) integers. Let $\mathcal{A}_+$ be the set of unbarred letters and $\mathcal{A}_-$ the set of barred letters so that $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$. We fix an ordering

$$1 < \overline{1} < 2 < \overline{2} < \cdots . \tag{6.1}$$

Definition 6.7. A skew partition is said to be a *horizontal strip* if its diagram does not contain a $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ and a vertical strip if it does not contain a $\begin{array}{|c|c|}\hline & \\ \hline\end{array}$. A *n-horizontal strip* (respectively *n-vertical strip*) is one that contains exactly n boxes.

Definition 6.8. A *super* SSYT of (skew) shape ν is a map $T : \lambda \to \mathcal{A}$ such that the labels $T(u)$ of cells u are weakly increasing in columns and rows and such that the sub-diagram of cells containing a certain unbarred (respectively, barred) letter a always forms a horizontal (respectively, vertical) strip. In other words there are no $\begin{array}{|c|}\hline a \\ \hline a \\ \hline\end{array}$ or $\begin{array}{|c|c|}\hline \overline{a} & \overline{a} \\ \hline\end{array}$ configurations. The set of such tableaux is denoted by $\mathsf{SSYT}_{\pm}(\nu)$.

Definition 6.9. Given a symmetric function f, we define its *superization* to be the symmetric function $\omega_Y f(X+Y)$. By ω_Y we mean that $f(X+Y)$ is interpreted as a symmetric function in the Y-variables.

This nomenclature is justified by the fact that substituting "super" tableaux for tableaux in the combinatorial definition of the Schur functions gives exactly the superization on the symmetric function side. We will show this now.

Notation 6.10. Let $Z = \{z_i \mid i \in \mathcal{A}\}$ and identify $X = \{x_i = z_i \mid \mathcal{A}_+\}$, $Y = \{y_i = z_i \mid \mathcal{A}_-\}$ so that $Z = X \cup Y$. For $T \in \mathsf{SSYT}_{\pm}(\nu)$, denote by z^T the monomial $\prod_{u \in \nu} z_{T(u)}$, which can thus be seen as a polynomial in X (unbarred labels) and Y (barred labels).

Proposition 6.11. For any partition λ, we have that

$$\omega_Y s_\lambda(X+Y) = \sum_{T \in \mathsf{SSYT}_\pm(\lambda)} z^T$$

Proof. Let us take a new ordering on $\mathcal{A}$: $1 < 2 < \cdots < \bar{1} < \bar{2} < \cdots$ Using this order in the definition of a super SSYT, is easy to see that

$$s_\lambda[X+Y] = \sum_{\mu \subseteq \lambda} s_\lambda[X] s_{\lambda/\mu}[Y]$$

Since $\omega s_{\lambda/\mu} = s_{(\lambda/\mu)'}$ (see Corollary 2.19), applying ω_Y gives

$$\omega_Y s_\lambda[X+Y] = \sum_{\mu \subseteq \lambda} s_\lambda[X] s_{(\lambda/\mu)'}[Y].$$

Given pair of tableaux $(T_1, T_2) \in \mathsf{SSYT}(\mu) \times \mathsf{SSYT}((\lambda/\mu)')$ construct a tableau in $\mathsf{SSYT}_\pm(\lambda)$ by transposing T_2, replacing all its labels by the same barred labels and placing it on top of T_1.

We can deduce the result for any ordering on $\mathcal{A}$ by using the symmetry s_λ.

Now let us define a "superization" of the fundamental quasisymmetric functions. Recall their definition.

Definition 3.4. For $S \subseteq [n-1]$, we define the *(Gessel) fundamental quasisymmetric function* of degree n indexed by S as

$$F_{n,S} = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ j \in S \implies i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Definition 6.12. For $S \subseteq [n-1]$

$$\tilde{F}_{n,S}(Z) = \tilde{F}_{n,S}(X,Y) = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \in \mathcal{A} \\ i_j = i_{j+1} \in \mathcal{A}_+ \Rightarrow j \notin S \\ i_j = i_{j+1} \in \mathcal{A}_- \Rightarrow j \in S}} z_{i_1} z_{i_2} \cdots z_{i_n}.$$

Proposition 6.13. For any schur function s_λ, we have

$$\omega_Y s_\lambda(X+Y) = \sum_{S \in SYT(\lambda)} \tilde{F}_{\mathsf{Des}(S)}(X,Y)$$

Proof. In view of Proposition 6.11 and Definition 6.12 we have to show that

$$\sum_{T \in \mathsf{SSYT}_\pm(\lambda)} z^T = \sum_{S \in \mathsf{SYT}(\lambda)} \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \in \mathcal{A} \\ i_j = i_{j+1} \in \mathcal{A}_+ \Rightarrow j \notin \mathsf{Des}(S) \\ i_j = i_{j+1} \in \mathcal{A}_- \Rightarrow j \in \mathsf{Des}(S)}} z_{i_1} z_{i_2} \cdots z_{i_n}.$$

Let us thus define a bijective map ϕ that associates to any $T \in \mathsf{SSYT}_\pm(\lambda)$ a pair (S, w), where $S \in \mathsf{SYT}(\lambda)$ and $w : [n] \to \mathcal{A}$ a such that $w(i) \le w(i+1)$ for all i; and such that $w(j) = w(j+1) \in \mathcal{A}_+$ implies that $j \notin \mathsf{Des}(S)$ and $w(j) = w(j+1) \in \mathcal{A}_-$ implies $j \notin \mathsf{Des}(S)$.

From T, define S to be the unique standard Young tableau that preserves the fixed ordering given on $\mathcal{A}$ (for example 6.1) and such that in the horizontal strips labelled a in T the labels are increasing from left to right in S and in the vertical strips labelled $\overline{a}$ in T the labels are increasing bottom to top in S. See Figure 6.3 for an example. Then set w to be $T \circ S^{-1}$, i.e. reading the labels of T in the ordering of the cells determined by S. For example, the weakly increasing word associated to super tableau on the left in Figure 6.3 is $11\overline{1}2\overline{2}\overline{2}\overline{2}33\overline{3}\overline{3}$. It is easy to see that by construction, equal unbarred letters do not create a descent in the super tableau and that equal barred letters do.

It is left to show that ϕ is bijective: we will describe its inverse. Given (S, w) as described above, we have $\phi^{-1}(S, w) = w \circ S$. Since the shape of a standard tableau with no descents must a horizontal strip and the shape of a standard tableau containing only descents must be a vertical strop, the resulting tableau is a super SSYT.

$$\begin{array}{|c|c|ccc}
\cline{1-2} \overline{2} & \overline{3} & & & \\
\cline{1-2} 2 & \overline{2} & & & \\
\cline{1-4} \multicolumn{1}{c|}{} & \overline{2} & 3 & \multicolumn{1}{c|}{3} & \\
\cline{2-5} \multicolumn{1}{c|}{} & 1 & 1 & \multicolumn{1}{c|}{\overline{1}} & \multicolumn{1}{c|}{\overline{3}} \\
\cline{2-5}
\end{array}
\qquad\qquad
\begin{array}{|c|c|ccc}
\cline{1-2} 7 & 11 & & & \\
\cline{1-2} 4 & 6 & & & \\
\cline{1-4} \multicolumn{1}{c|}{} & 5 & 8 & \multicolumn{1}{c|}{9} & \\
\cline{2-5} \multicolumn{1}{c|}{} & 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{10} \\
\cline{2-5}
\end{array}$$

Figure 6.3: A super SSYT (left) and its standardisation (right).

Notice that taking $Y = 0$ we get Proposition 3.8.

Let us extend this result to the whole symmetric function ring.

Corollary 6.14. Let f be any symmetric function, homogeneous of degree n such that

$$f = \sum_{D \subseteq [n-1]} c_D F_{n,D}$$

then

$$\omega_Y f(X + Y) = \sum_{D \subseteq [n-1]} c_D \tilde{F}_{n,D}(X, Y).$$

Proof. It follows by expanding f into the Schur basis, Proposition 6.13 and the linear independence of the $F_{n,D}$.

6.2.2 Super LLT's and fundamental expansion

Let us define a superization of LLT polynomials. In what follows we take $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(k)})$ to be a tuple of skew partitions,

Definition 6.15. The set of *super SSYT-tuples* is defined as

$$\mathsf{SSYT}_{\pm}(\boldsymbol{\nu}) \coloneqq \mathsf{SSYT}_{\pm}(\nu^{(1)}) \times \cdots \times \mathsf{SSYT}_{\pm}(\nu^{(k)}).$$

See for example Figure 6.4 (left).

Definition 6.16. Given $T \in \mathsf{SSYT}_{\pm}(\boldsymbol{\nu})$, a pair of cells (u, v) with $u \in \nu^{(i)}$ and $v \in \nu^{(j)}$ and $i < j$ is called an *inversion pair* if one of the following holds

- $c(u) = c(v)$ and $(T_i(u) > T_j(v)$ or $T_i(u) = T_j(v) \in \mathcal{A}_-)$;
- $c(u) + 1 = c(v)$ and $(T_i(u) < T_j(v)$ or $T_i(u) = T_j(v) \in \mathcal{A}_-)$.

Definition 6.17. The *reading order* of $\boldsymbol{\nu}$ is a total order $<_r$ on the cells of $\sqcup_{i\in[k]}\nu_i$ defined as follows: for $u \in \nu_i$ and $v \in \nu_j$, we say that $u <_r v$ if

- either $c(u) > c(v)$;
- or $c(u) = c(v)$ and $i < j$;
- or $c(u) = c(v)$, $i = j$ and u is above and to the right of v in $\nu_i = \nu_j$.

In other words, follow the diagonals from highest to lowest, from top right to bottom left.

Definition 6.18. Define the *super LLT polynomial* to be

$$\widetilde{LLT}_{\boldsymbol{\nu}}(Z) = \widetilde{LLT}_{\boldsymbol{\nu}}(X, Y) \coloneqq \sum_{\mathbf{T}\in\mathsf{SSYT}_\pm(\boldsymbol{\nu})} q^{\mathsf{inv}(\mathbf{T})} z^T$$

Definition 6.19. The *reading word* of an SSYT-tuple $\mathbf{T} \in \mathsf{SSYT}_\pm(\boldsymbol{\nu})$ is the word obtained by reading the labels $\mathbf{T}(u)$ in the reading order of $\boldsymbol{\nu}$.

For example, the reading word of the SSYT-tuple in Figure 6.1 is 6455513313.

Definition 6.20. Given an SYT-tuple $T \in \mathsf{SYT}(\boldsymbol{\nu})$, define its *descent set*

$$\mathsf{Des}(\mathbf{T}) \coloneqq \{i \mid i + 1 \text{ precedes } i \text{ in } \mathbf{T}\text{'s reading word}\}.$$

Proposition 6.21. We have

$$\widetilde{LLT}_{\boldsymbol{\nu}}(X, Y) = \sum_{\mathbf{T}\in\mathsf{SYT}(\boldsymbol{\nu})} q^{\mathsf{inv}(\mathbf{T})} \tilde{F}_{\mathsf{Des}(\mathbf{T})}.$$

Proof. The proof is similar to the proof of Proposition 6.13. Given a super SSYT tuple $\mathbf{T}$, define its standardization $\mathbf{S}$ to be the unique SYT tuple that preserves the fixed ordering on $\mathcal{A}$ (for example 6.1) and such that the $\mathbf{T}^{-1}(\{a\})$ are ordered increasingly with respect to the reading order and the $\mathbf{T}^{-1}(\{\overline{a}\})$ are ordered decreasingly with respect to the reading order. Notice that this standardization conserves the inv.

For example, Figure 6.4 shows a super SSYT tuple and its standardization.

Setting $Y = 0$ we get the expansion of LLT polynomials in the fundamentals.

Corollary 6.22. We have

$$LLT_{\boldsymbol{\nu}}(X) = \sum_{\mathbf{T}\in\mathsf{SYT}(\boldsymbol{\nu})} q^{\mathsf{inv}(\mathbf{T})} F_{\mathsf{Des}(\mathbf{T})}.$$

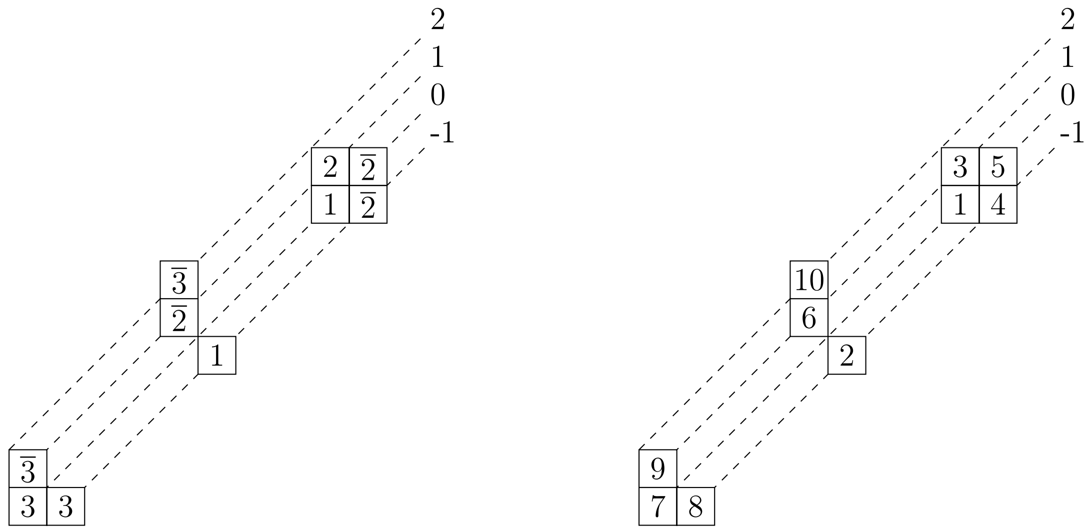


Figure 6.4: A super SSYT (left) and its standardization (right)

6.2.3 The proof

We finally prove Theorem 6.6. The following definition will be handy in what is to come.

Definition 6.23. For $u \in \nu^{(i)}$ and $v \in \nu^{(j)}$ with $i < j$, we say that u *and* v *attack each other* if either $c(u) = c(v)$ or $c(u) + 1 = c(v)$. We may also say that u and v form an *attack relation*

Let $\boldsymbol{\nu}'$ be obtained from $\boldsymbol{\nu}$ by transposing each $\nu^{(i)}$ and reversing the order of the tuple.

Lemma 6.24. $LLT_{\boldsymbol{\nu}}$ is symmetric if and only if $LLT_{\boldsymbol{\nu}'}$ is.

Proof. Suppose $LLT_{\boldsymbol{\nu}}$ is symmetric. Then, in view of Corollary 6.14, we may conclude that $\widetilde{LLT}_{\boldsymbol{\nu}}(X,Y) = \omega_Y LLT(X+Y)$ and so $\widetilde{LLT}_{\boldsymbol{\nu}}(X,Y)$ is symmetric in both the X and Y variables. So $\widetilde{LLT}_{\boldsymbol{\nu}}(0,Y)$, which is the generating function of super SSYT tuples with only barred letters, is symmetric. Take $\mathbf{T}$ such a tableau and define $\mathbf{T}'$ to be the SSYT tuple obtained from T by removing the bar from all the letters, transposing each tableaux, and reversing the order of the tuple. See Figure 6.5. Clearly $\mathbf{T}'$ is an SSYT tuple of shape $\boldsymbol{\nu}'$. If m is the number of attack relations of $\boldsymbol{\nu}$, then it is also the number of attack relations of $\boldsymbol{\nu}'$ and it is easy to see that if u attacks v in $\boldsymbol{\nu}'$ then (u,v) is an inversion in $\boldsymbol{\nu}$ if and only if (u',v') is not an inversion in $\boldsymbol{\nu}'$. Thus we have $q^m \widetilde{LLT}_{\boldsymbol{\nu}}(0,Y)_{|q=-1} = LLT_{\boldsymbol{\nu}'}(Y)$ and so $LLT_{\boldsymbol{\nu}'}$ is symmetric.

We make a series of reductions.

- It suffices to show that $LLT_{\boldsymbol{\nu}}$ is symmetric in x_i and x_{i+1} for all i.

- Given a $\mathbf{T} \in \mathsf{SSYT}(\boldsymbol{\nu})$, let S be the sub-SSYT tuple containing either labels i or $i+1$. We notice that any attack relation with an element from S and an element from outside S will not be influenced by a re-labelling of S with the alphabet $\{i, i+1\}$. Thus we may restrict our attention to tableaux with only two distinct labels, say 1 and 2.

- For $\mathbf{T} \in \mathsf{SSYT}(\boldsymbol{\nu})$ with only 2 distinct labels, each tableau of the tuple has at most two cells in a column. Consider such a pair of cells, $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$, call the bottom cell u and the top cell v.

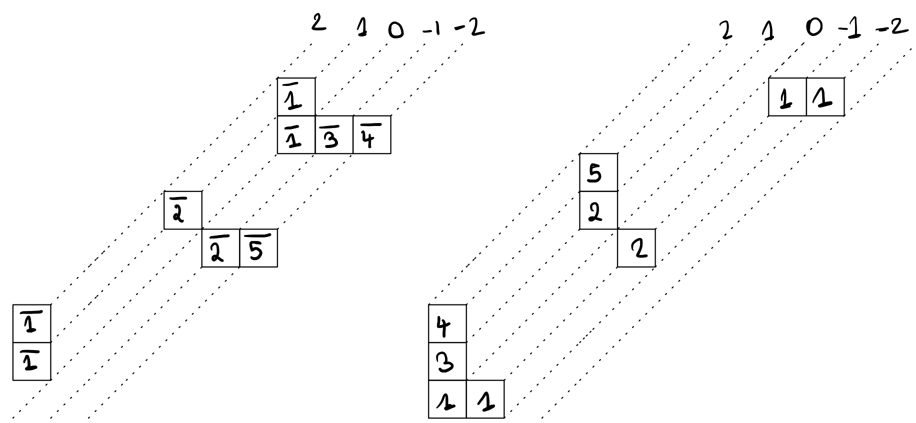


Figure 6.5: A super SSYT $\mathbf{T}$ with only barred labels (left) and its corresponding SSYT (right).

Now consider a third cell w that forms an attack relation with either u or v. It is easy to see that changing the label of w from a 1 to a 2 or vice versa conserves the total amount of inv created between u, v and w. It follows that we can disregard all two-cell columns and restrict our attention to tuples of horizontal strips.

- Using Lemma 6.24 and the previous point, we may restrict our argument to tuples that are both horizontal and vertical strips. In other words each $\nu^{(i)}$ is a disconnected union of single cells.

So let

$$G_{\boldsymbol{\nu}}(X) = \sum_{\substack{\mathbf{T}\in\mathsf{SSYT}(\boldsymbol{\nu}) \\ \nu^{(i)} \text{ vertical and horizontal strip} \\ \mathrm{Im}(\mathbf{T})=\{1,2\}}} q^{\mathsf{inv}(\mathbf{T})} x^{\mathbf{T}}$$

By the above reductions, it suffices to show the symmetry of G. Take $n = \sum_i |\nu^{(i)}|$ be the total number of cells and $u_1 \cdots u_n$ be the word reflecting the reading order of these cells. Set r to be the number of cells that form an attack relation with u_n. We will proceed by induction on n and r.

If $n = 0$, we have $G_\emptyset = 1$.

If $r = 0$ then u_n has no attack relations with the rest of the tuple, so there are no inv pairs involving u_n. Let $\boldsymbol{\nu}'$ be the tuple obtained from $\boldsymbol{\nu}$ by deleting u_n. Then we have $G_{\boldsymbol{\nu}} = (x_1 + x_2)G_{\boldsymbol{\nu}'}$ and this right hand side is symmetric by the induction hypothesis.

If $r > 0$, let $\boldsymbol{\nu}'$ be the tuple obtained from $\boldsymbol{\nu}$ by moving u_n such that it will have exactly one less attack relation: see Figure 6.6. Let $\mathbf{T} \in \mathsf{SSYT}(\boldsymbol{\nu})$ and $\mathbf{T}' \in \mathsf{SSYT}(\boldsymbol{\nu}')$. Then we have

$$\mathsf{inv}(\mathbf{T}) = \mathsf{inv}(\mathbf{T}') + \chi(T(u_n) = 1 \text{ and } T(u_{n-r}) = 2)$$

Thus $G_{\boldsymbol{\nu}} - G_{\boldsymbol{\nu}'}$ gives the generating function the tableaux tuples of shape $\boldsymbol{\nu}$ with $\mathbf{T}(u_n) = 1$ and $\mathbf{T}(u_{n-r}) = 2$, minus the generating function of the tableaux tuples of shape $\boldsymbol{\nu}'$ with $\mathbf{T}'(u_n) = 1$ and $\mathbf{T}'(u_{n-r}) = 2$.

If $\mathbf{T} \in \mathsf{SSYT}(\boldsymbol{\nu})$, the number of inv pairs involving u_{n-r} and u_n equals exactly r, indeed no u_i for $i < n-r$ creates inv with either u_{n-r} nor u_n, (u_{n-r}, u_n) is an inv pair and each u_i for $n-r < i < n$ creates inv with either u_n or u_{n-r}.

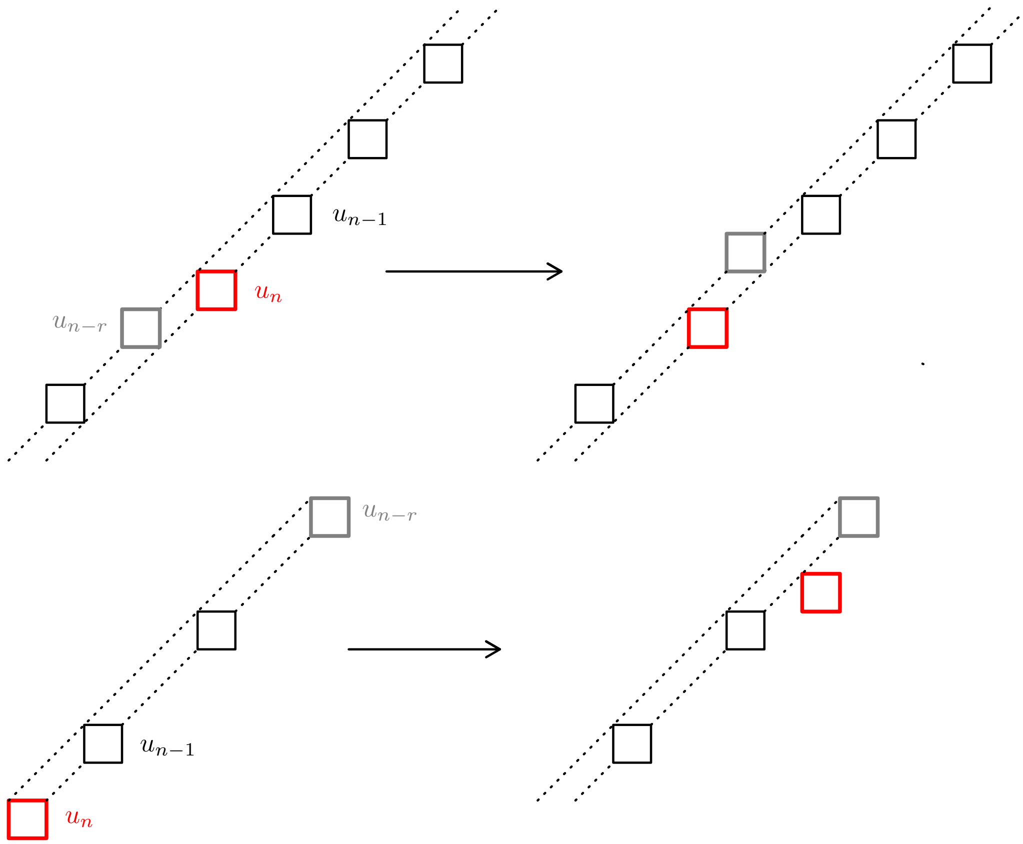


Figure 6.6: Schematic representation of the transformation $\boldsymbol{\nu} \mapsto \boldsymbol{\nu}'$.

Similarly, if $\mathbf{T}' \in \mathsf{SSYT}(\boldsymbol{\nu}')$ is such that $\mathbf{T}'(u_n) = 1$ and $\mathbf{T}'(u_{n-r}) = 2$, the number of inv pairs involving u_{n-r} and u_n equals exactly $r - 1$ by construction.

It follows that if $\boldsymbol{\kappa}$ be the tuple obtained from $\boldsymbol{\nu}$ by deleting u_{n-r} and u_n we have

$$G_{\boldsymbol{\nu}} - G_{\boldsymbol{\nu}'} = (q^r - q^{r-1})x_1 x_2 G_{\boldsymbol{\kappa}}.$$

Since $G_{\boldsymbol{\nu}'}$ is symmetric by induction on r and $G_{\boldsymbol{\kappa}}$ by induction on n, we may conclude.

Chapter 7

e-expansion at $q = q + 1$ of the unicellular LLT

[AS22]

Chapter 8

Unicellular LLT's and chromatic symmetric functions

The main result of this course is a plethystic relation between the chromatic (quasi)-symmetric functions obtained from the incomparability graph of a $3+1$ and $2+2$-free poset by taking all the possible colorings and only the proper ones.

Definition 8.1. Let G be a graph with $V = [n]$. Define

$$\overline{\mathcal{X}}_G(X, q) \coloneqq \sum_{\kappa \text{ coloring}} q^{\mathsf{asc}(\kappa)} x^{\kappa}.$$

We have the following.

Theorem 8.2. Let G be the incomparability graph of a $3+1$ and $2+2$-free poset. Then

$$\overline{\mathcal{X}}_G[X(q-1), q] = (q-1)^{\# V} \mathcal{X}_G[X, q].$$

Proof. Recall that $\overline{\mathcal{X}}_G[X, q]$ is a unicellular LLT-polynomial, which we can interpret as

$$\overline{\mathcal{X}}_G[X, q] = \sum_{w \in \mathbb{N}_+^n} q^{\mathsf{inv}(D_G, w)} x^w,$$

where D_G is the Dyck path such that $G_D = D_G$.

By [CM18, Proposition 3.5] (see also [HX17, Proposition 3.5], or [HX17, Appendix] for an elementary proof), the plethystic evaluation at $X(q-1)$, up to the normalization by $(q-1)^n$, corresponds to selecting the paths with no attack relation between cells with the same labels, which correspond exactly to proper colorings. The thesis follows.

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