Combinatorics of the Delta conjecture at $q=-1$

ANNA VANDEN WYNGAERD

joint work with SYLVIE CORTEEL

MATTHIEU JOSUAT-VERGÈS

STARTING POINT

$$\langle \nabla e_n , h_1^n \rangle \Big|_{\substack{q=-1 \\ t=1}}$$

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$n=1$ $\qquad$ 1

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$n = 1$ 1

$n = 2$ 1

STARTING POINT

$$\langle \nabla e_n , h_1^n \rangle \Big|_{\substack{q=-1 \\ t=1}}$$

$n=1$ $\quad$ 1

$n=2$ $\quad$ 1

$n=3$ $\quad$ 2

STARTING POINT

$$\langle \nabla e_n , h_1^n \rangle \Big|_{\substack{q=-1\\ t=1}}$$

$n=1$	1
$n=2$	1
$n=3$	2
$n=4$	5

STARTING POINT

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$n=1$	1
$n=2$	1
$n=3$	2
$n=4$	5
$n=5$	16

STARTING POINT

$$\langle \nabla e_n , h_1^n \rangle \Big|_{\substack{q=-1 \\ t=1}}$$

$n=1$	1
$n=2$	1
$n=3$	2
$n=4$	5
$n=5$	16
$n=6$	61

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$n=1$	1
$n=2$	1
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$n=6$	61
$n=7$	272

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$n=9$	7936

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$$\left\langle \mathrm{Den}_n ,\ h_1^n \right\rangle \Big|_{\substack{q=-1 \\ t=1}}$$

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$n=9$	7936

THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

1 1 2 5 16 61 272 1385 7936 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,1,2,5,16,61,272,1385,7936**

Displaying 1-5 of 5 results found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A000111 Euler or up/down numbers: e.g.f. sec(x) + tan(x). Also for n >= 2, half the number of alternating permutations on n letters (A001250). +30 319

(Formerly M1492 N0587)

1, **1, 1, 2, 5, 16, 61, 272, 1385, 7936**, 50521, 353792, 2702765, 22368256, 199360981, 1903757312, 19391512145, 209865342976, 2404879675441, 29088885112832, 370371188237525, 4951498053124096, 69348874393137901, 1015423886506852352, 15514534163557086905, 246921480190207983616, 4087072509293123892361 (list; graph; refs; listen; history; text; internal format)

WHAT is ∇e_n?

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$\Lambda_{\mathbb{Q}(q,t)}$ = The space of formal power series (of bounded degree) in an infinite set of variables $z_1, z_2, z_3, \ldots$ with coefficients in $\mathbb{Q}(q,t)$ and invariant by all permutations of these z-variables

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$$e_n = \sum_{i_1 < \cdots < i_n} z_{i_1} \cdots z_{i_n}$$

eg $e_2 = z_1 z_2 + z_1 z_3 + z_2 z_3 + \cdots$

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THM $\Lambda_{\mathbb{Q}(q,t)} = \mathbb{Q}(q,t)[e_1, e_2, \ldots]$

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THM $\Lambda_{\mathbb{Q}(q,t)} = \mathbb{Q}(q,t)[e_1, e_2, \ldots]$

∇ = Some operator on $\Lambda_{\mathbb{Q}(q,t)}$ whose eigenvectors are Macdonald polynomials (*)

[(*) Bergeron - Garsia - 1999]

A REASON TO CARE ABOUT ∇e_n

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The Frobenius characteristic map

$$\mathcal{F} : \mathfrak{S}_n\text{-representations} \longrightarrow \Lambda_{\mathbb{C}}^{(n)}$$

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The Frobenius characteristic map

$$F : S_n\text{-representations} \longrightarrow \Lambda^{(n)}_{\mathbb{C}}$$

homogeneous degree n

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$$\text{Irreducibles} \longmapsto \text{Schur functions}$$

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The Frobenius characteristic map

$$\mathcal{F}_{qt} : \text{Bi-graded } S_n\text{-representations} \longrightarrow \Lambda^{(n)}_{\mathbb{C}(q,t)}$$

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Diagonal coinvariants $\qquad R_n = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$

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$\mathfrak{S}_n$

ACTS "DIAGONALLY"

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$$\boxed{\textbf{THM}^{(*)} \quad \mathcal{F}_{q,t}(\mathcal{DH}_n) = \nabla e_n}$$

[(*) Haiman 2002]

A REPRESENATION-THEORETIC REASON TO CARE ABOUT $q=-1$

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... probably ?

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$$A = \mathbb{C}[X^1, \dots, X^{k_1}, \Theta^1, \dots, \Theta^{k_2}] / (\quad)^{S_n}_{+}$$

$$\mathcal{F}(A) = \sum c_{\lambda\mu} \, s_\mu[Q - \varepsilon U] \, s_\lambda[Z] \quad (*)$$

(*) Bergeron 2020

THE COMBINATORICS OF $\mathcal{D}_n$

THE COMBINATORICS OF ∇e_n

SHUFFLE THM (*) (□)

$$\nabla e_n = \sum_{D \in LD(n)} q^{\mathrm{dinv}(D)}\, t^{\mathrm{area}(D)}\, z^D$$

(*) Conjecture [Haglund - Haiman - Loehr - Remmel - Ulyanov - 2005]

(□) Proof [Carlsson - Mellit - 2016]

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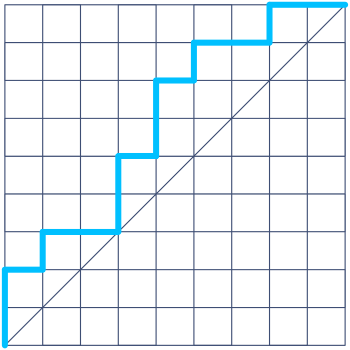

Dyck path

(*) Conjecture [Haglund - Haiman - Loehr - Remmel - Ulyanov - 2005]
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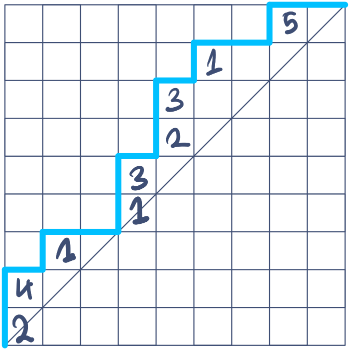

Dyck path

labels strictly increasing in columns ↑

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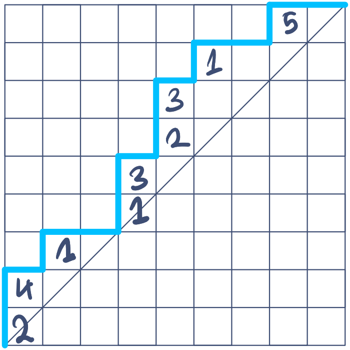

Dyck path

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$$z^D = z_1^3 z_2^2 z_3^2 z_4 z_5$$

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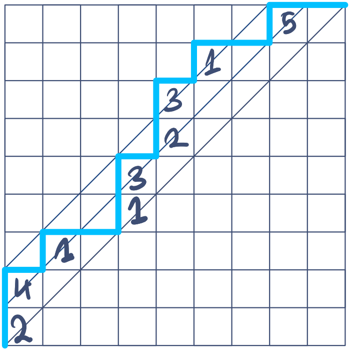

Dinv (= Diagonal inversions)

(*) Conjecture [Haglund - Haiman - Loehr - Remmel - Ulyanov - 2005]
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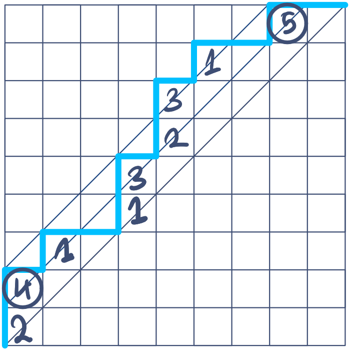

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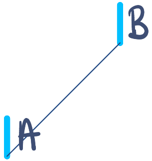

$A < B$

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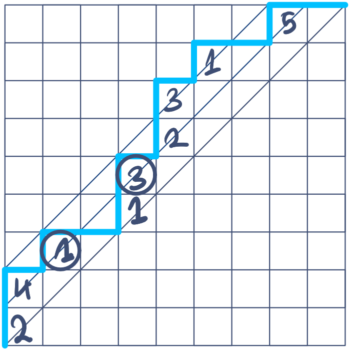

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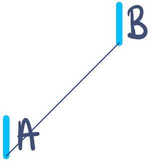

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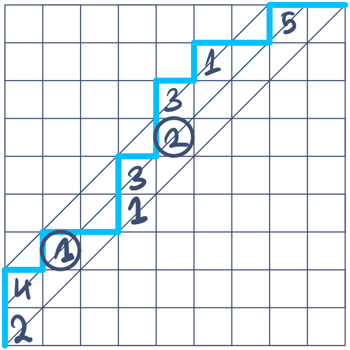

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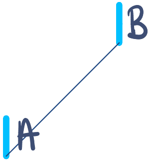

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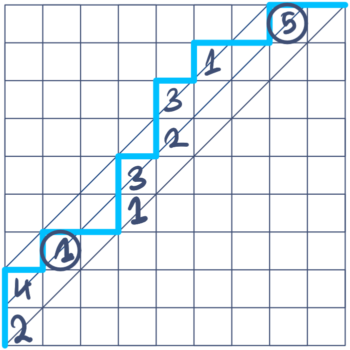

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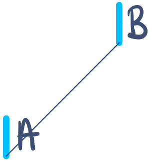

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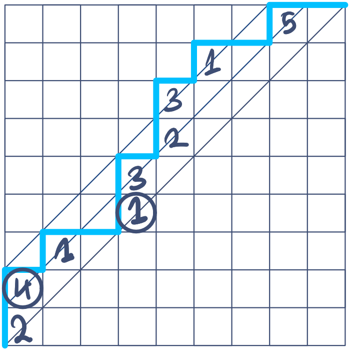
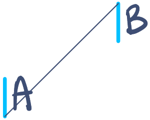


Dinv (= Diagonal inversions)

Secondary



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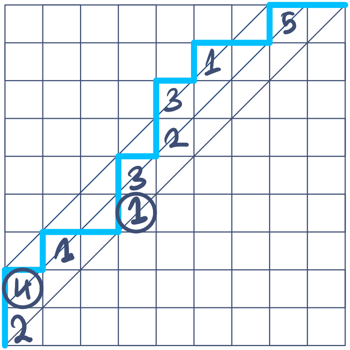
$$\nabla e_n = \sum_{D \in LD(n)} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)} x^D$$



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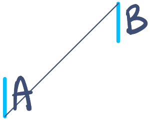

→ 4 units of primary dinv

→ 1 unit of secondary dinv

⇒ dinv = 4

(*) Conjecture [Haglund - Haiman - Loehr - Remmel - Ulyanov - 2005]

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THE COMBINATORICS OF ∇e_n

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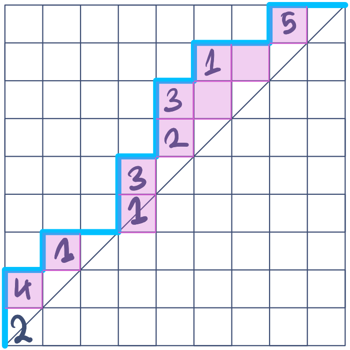

AREA $= 10$

(*) Conjecture [Haglund - Haiman - Loehr - Remmel - Ulyanov - 2005]
(□) Proof [Carlsson - Mellit - 2016]

THE COMBINATORICS OF ∇e_n

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$\langle \nabla e_n , e_n \rangle$ = qt-Catalan numbers (*)

(*) Garsia-Haglund 2001-2002

THE COMBINATORICS OF ∇e_n

$\langle \nabla e_n, e_n \rangle$ = qt- Catalan numbers (*)

$\langle \nabla e_n, e_n h_{n-d} \rangle$ = qt- Schröder numbers ($\square$)

(*) Garsia - Haglund 2001-2002
($\square$) Haglund 2004

THE COMBINATORICS OF ∇e_n

$\langle \nabla e_n, e_n \rangle$ = qt-Catalan numbers (∗)

$\langle \nabla e_n, e_n h_{n-d} \rangle$ = qt-Schröder numbers (□)

$\langle \nabla e_n, h_1^n \rangle$ = qt-parking functions (≈)

(∗) Garsia-Haglund 2001-2002 (≈) Carlsson-Mellit 2016
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THE COMBINATORICS OF ∇e_n

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$\langle \nabla e_n, h_1^n \rangle \big|_{q=0}$ = $[n]_t!$ [inversions in permutations]

(*) Garsia-Haglund 2001-2002 (≈) Carlsson-Mellit 2016

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THE COMBINATORICS OF ∇e_n

$\langle \nabla e_n, e_n \rangle$ = qt- Catalan numbers (*)

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$\langle \nabla e_n, h_1^n \rangle \big|_{q=0} = [n]_t!$ [inversions in permutations]

THIS TALK

$$\langle \nabla e_n, h_1^n \rangle \big|_{q=-1} = t\text{- Euler numbers}$$

(*) Garsia-Haglund 2001-2002 (≈) Carlsson-Mellit 2016
(□) Haglund 2004

A t-ANALOGUE FOR EULER NUMBERS

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Alternating permutations $\sigma \in \mathfrak{S}_n$ such that $\sigma_1 > \sigma_2 < \sigma_3 > \cdots$

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Counted by Euler numbers E_n : $\tan(x) + \sec(x) = \sum_{n \geq 0} E_n \frac{x^n}{n!}$

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ex $n=4$

4 1 3 2
4 2 3 1
3 1 4 2
3 2 4 1
2 1 4 3

A t-ANALOGUE FOR EULER NUMBERS

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A 31-2 pattern of $\sigma \in \mathfrak{S}_n$ is a pair $1 \leq i < j \leq n$ such that $\sigma_{i+1} < \sigma_j < \sigma_i$

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ex $n=4$

4 1 3 2	$\rightarrow$	2
4 2 3 1	$\rightarrow$	1
3 1 4 2	$\rightarrow$	1
3 2 4 1	$\rightarrow$	0
2 1 4 3	$\rightarrow$	0

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ex $n=4$

4 1 3 2	$\to$	2
4 2 3 1	$\to$	1
3 1 4 2	$\to$	1
3 2 4 1	$\to$	0
2 1 4 3	$\to$	0

$$\boxed{E_n(t) = \sum_{\sigma \in \mathcal{A}_n} t^{\#31\text{-}2(\sigma)} \quad (*)\ (\square)\ (\approx)}$$

$$E_4(t) = t^2 + 2t + 2$$

A 31-2 pattern of $\sigma \in \mathfrak{S}_n$ is a pair $1 \leq i < j \leq n$ such that $\sigma_{i+1} < \sigma_j < \sigma_i$

(*) Han-Randrianarivony-Zeng 1999 ($\square$) Chebikin - 2008 ($\approx$) Josuat-Vergès 2010

THM (Corteel, Josuat-Vergès, VW)

$$\langle \nabla e_n, h_1^n \rangle \Big|_{q=-1} = t^{\lfloor \frac{n^2}{4} \rfloor} E_n(t)$$

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The proof: Zoom out to see a clearer picture!

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RHS $\quad$ Alternating permutations $\rightsquigarrow$ Permutations
31-2 avoidance $\quad$ mahonian statistics

Inspired by

$$(*) \quad \sum_{k=0}^{n-1} \langle \Delta'_{e_{n-k-1}} e_n, h_1^n \rangle\big|_{q=-1} = [n]_t!$$

(*) D'Adderio - Iraci - VW 2020

MAHONIAN STATISTICS ON PERMUTATIONS

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A Mahonian statistic on permutations is a map, $\mathrm{stat}: \mathfrak{S}_n \longrightarrow \mathbb{N}$ such that

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$\mathfrak{S}_3$	maj
123	0
132	2
213	1
231	2
312	1
321	3

$$(1+t+t^2)(1+t) = 1+2t+2t^2+t^3$$

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Inv number of inversion pairs : $\#\{\, i<j \mid \sigma_i > \sigma_j \,\}$

$\mathfrak{S}_3$	maj	inv
123	0	0
132	2	1
213	1	1
231	2	2
312	1	2
321	3	3

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(*) Alternating inversion $\#\{i<j \mid i \text{ odd}, \sigma_i > \sigma_j\} + \#\{i<j \mid i \text{ even } \sigma_i < \sigma_j\}$

$\mathfrak{S}_3$	maj	inv	$\hat{\imath}$
123	0	0	1
132	2	1	0
213	1	1	2
231	2	2	1
312	1	2	3
321	3	3	2

(*) Chebikin 2008

MAHONIAN STATISTICS ON PERMUTATIONS

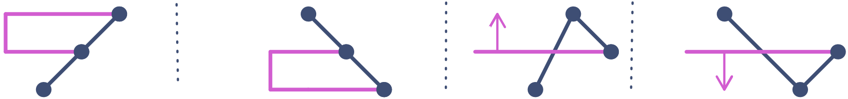
(*) **Inv_3** Given $\sigma \in \mathfrak{S}_n$ (set $\sigma_0 = 0$) a pair $i < j$ is a 3-inversion if

(1) σ_j double ascent and $\sigma_{j-1} < \sigma_i < \sigma_j$

(2) σ_j double descent and $\sigma_{j-1} > \sigma_i > \sigma_j$

(3) σ_j peak and $\sigma_i > \sigma_j$

(4) σ_j valley and $\sigma_i < \sigma_j$



$\mathfrak{S}_3$	maj	inv	$\hat{\imath}$
1 2 3	0	0	1
1 3 2	2	1	0
2 1 3	1	1	2
2 3 1	2	2	1
3 1 2	1	2	3
3 2 1	3	3	2

Given $\sigma \in \mathfrak{S}_n$ (set $\sigma_0 = 0$) and $i \in \{2, \dots, n\}$

- σ_i is a double ascent if
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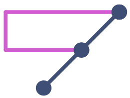
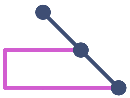


(2) σ_j double descent and $\sigma_{j-1} > \sigma_i > \sigma_j$



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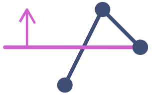

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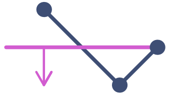

$\mathfrak{S}_3$	maj	inv	$\hat{\imath}$	inv_3
1 2 3	0	0	1	0
1 3 2	2	1	0	1
2 1 3	1	1	2	3
2 3 1	2	2	1	2
3 1 2	1	2	3	2
3 2 1	3	3	2	1

Given $\sigma \in \mathfrak{S}_n$ (set $\sigma_0 = 0$) and $i \in \{2, \ldots, n\}$

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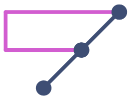

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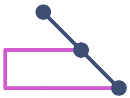

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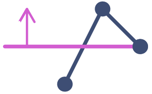

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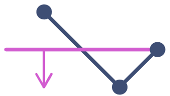

$\mathfrak{S}_3$	maj	inv	$\hat{\imath}$	inv_3
123	0	0	1	0
132	2	1	0	1
213	1	1	2	3
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Mahonian?

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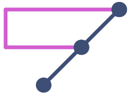

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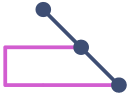

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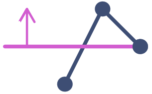

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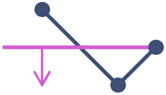

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ALTERNATING 'INVERIONS', INV_3 AND t-EULER NUMBERS

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(*) Chebikin 2008

ALTERNATING INVERSIONS, INV3 AND t-EULER NUMBERS

For $\sigma \in A_n$ we have $inv_3(\sigma) = \hat{\imath}(\sigma)$.

$$\boxed{\text{PROP (*)} \quad \sum_{\sigma \in A_n} t^{\hat{\imath}(\sigma)} = t^{\lfloor \frac{n^2}{4} \rfloor} E_n(t)}$$

So we have

$$\sum_{\sigma \in A_n} t^{inv_3(\sigma)} = t^{\lfloor \frac{n^2}{4} \rfloor} E_n(t)$$

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THE VALLEY DELTA CONJECTURE

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CONJ (*)

$$\Delta'_{e_{n-k-1}} e_n = \sum_{D \in LD(n)^{\bullet k}} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)} z^D$$

(*) Haglund - Remmel - Wilson 2018

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Δ'_f = another operator with Macdonald eigenvectors
"reduces" to ∇ when $k=0$

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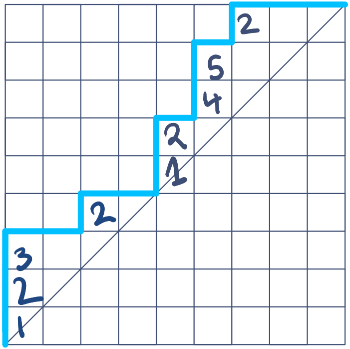

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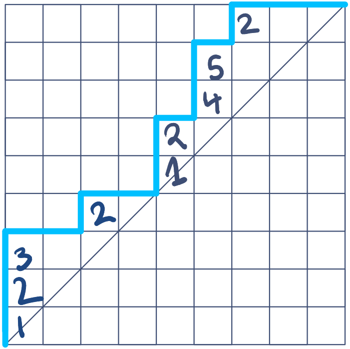
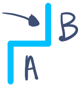


Choose k contractible valleys to decorate with a $\bullet$



A<B

(*) Haglund-Remmel-Wilson 2018

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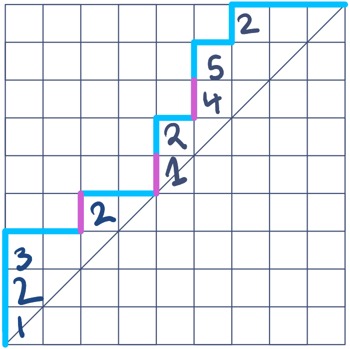

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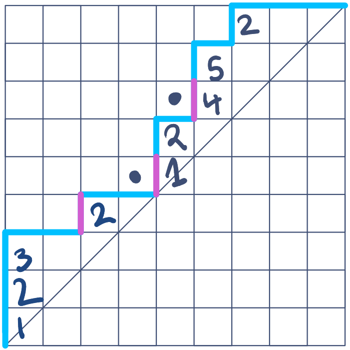

Choose k contractible valleys to decorate with a •

A<B

k=2

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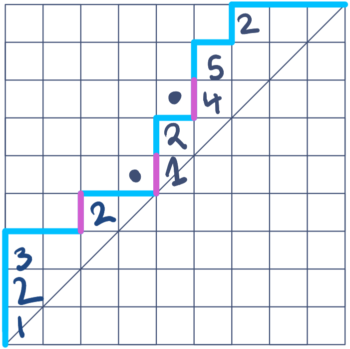

Choose k contractible valleys to decorate with a •

A<B

k=2

Influence on dinv

- disregard dinv pairs where left step is decorated
- -1 for each decoration

(*) Haglund-Remmel-Wilson 2018

What happens to the combinatorics of the shuffle theorem/Delta conjecture when setting $q=-1$?

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THM (Corteel, Josuat-Vergès, VW)

$$\sum_{k=0}^{n-1} u^k \sum_{D \in \mathrm{sTLD}(n)^{\bullet k}} (-1)^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}_3(\sigma)} u^{\mathrm{mond}(\sigma)}$$

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$\mathrm{monot}(\sigma) = \#$ double ascents/descents

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What happens to the combinatorics of the shuffle theorem/ Delta conjecture when setting $q=-1$?

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$$\underbrace{\sum_{k=0}^{n-1} u^k \sum_{D \in \mathrm{STLD}(n)^{\bullet k}} (-1)^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)}}_{\text{Delta conjecture}} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}_3(\sigma)} u^{\mathrm{monot}(\sigma)}$$

$\langle \Delta'_{e_{n-k-1}} e_n , h_1^n \rangle \big|_{q=-1}$

$\mathrm{monot}(\sigma) = \#$ double ascents / descents

$\Rightarrow u=0$ gives alternating permutations

We need schedule numbers

SCHEDULE NUMBERS

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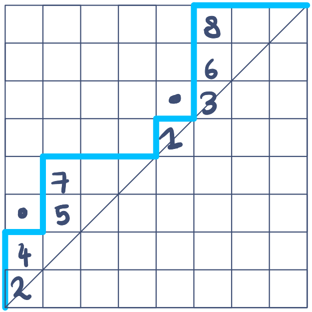
$$\langle \Delta'_{e_{n-k-1}} e_n , h_1^n \rangle = \sum_{D \in stLD(n)^{\ast k}} q^{dinv(D)} t^{area(D)}$$



SCHEDULE NUMBERS

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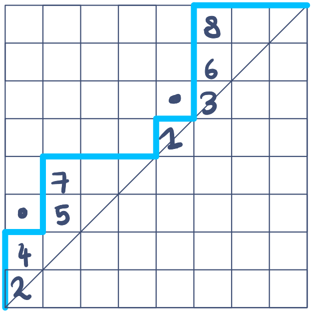
Schedule numbers provide a (combinatorially constructive) factorization of this sum (*) (□)



(*) Hicks 2013 (□) Haglund-Sergel 2021

SCHEDULE NUMBERS

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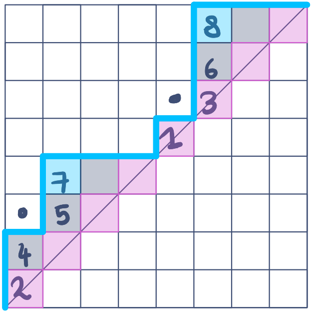

(decorated) diagonal word of a path $D \in LD(n)^{*k}$: $dw(D) = \dot{3}21 \; 6\dot{5}4 \; 87$

(∗) Hicks 2013 (□) Haglund-Sergel 2021

SCHEDULE NUMBERS

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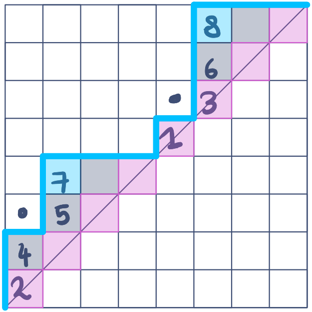

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(decorated) diagonal word of a path $D \in \mathrm{LD}(n)^{*k}$: $dw(D) = \dot{3}21 \; 6\dot{5}4 \; 87$

Schedule number assigns to each letter of the diagonal word τ an integer value $S_i(\tau)$

$$\sum_{\substack{D \in \mathrm{stLD}(n)^{*k} \\ dw(D) = \tau}} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)} = t^{\mathrm{revmaj}(\tau)} \prod_{i=1}^{n} [S_i(\tau)]_q$$

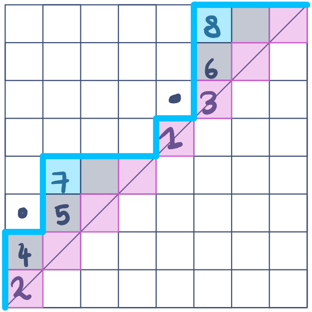
(□)

(*) Hicks 2013 (□) Haglund-Sergel 2021

SCHEDULE NUMBERS

$$\langle \Delta'_{e_{n-k-1}} e_n , h_1^n \rangle = \sum_{D \in \mathrm{stLD}(n)^{\bullet k}} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)}$$

Schedule numbers provide a combinatorially constructive) factorization of this sum (*) (□)



(decorated) diagonal word of a path $D \in \mathrm{LD}(n)^{\bullet k}$: $dw(D) = \dot{3}21\ 6\dot{5}4\ 87$

Schedule number assigns to each letter of the diagonal word τ an integer value $S_i(\tau)$

$$\sum_{\substack{D \in \mathrm{stLD}(n)^{\bullet k} \\ dw(D) = \tau}} q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)} = t^{\mathrm{revmaj}(\tau)} \prod_{i=1}^{n} [S_i(\tau)]_q$$

(□)

$\mathrm{revmaj}(\tau) = \mathrm{maj}(\tau^{\mathrm{rev}})$

MAHONIAN!

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SCHEDULE NUMBERS : EXAMPLE

Take $\tau = 2\ \dot{1}\ \ 4\ 3$

$s = 1\ 2\ \ 1\ 2$

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$$\emptyset \xrightarrow{1} \boxed{2} \xrightarrow{1} \begin{array}{|c|c|}\hline 4 & \\ \hline 2 & \\ \hline\end{array}$$

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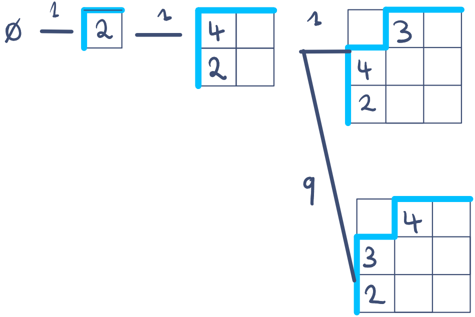

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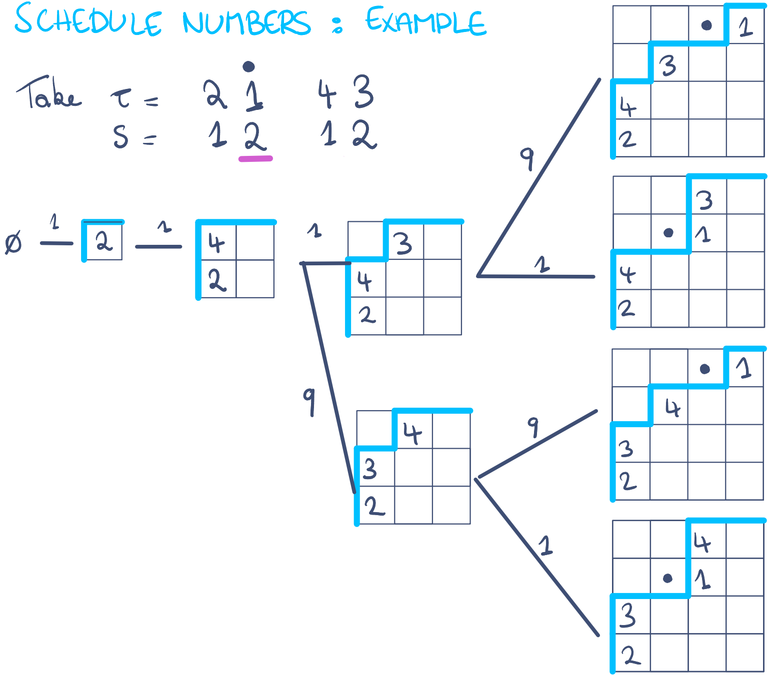

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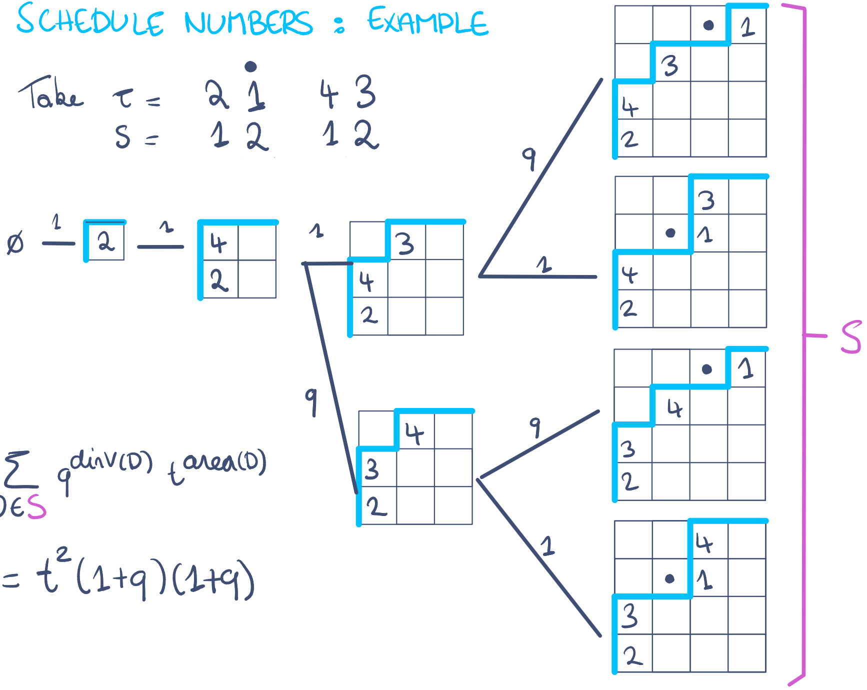

$$\sum_{D \in S} q^{\mathrm{dinv}(D)}\, t^{\mathrm{area}(D)}$$

$$= t^2(1+q)(1+q)$$

SCHEDULE NUMBERS AT $q = -1$

SCHEDULE NUMBERS AT $q=-1$

LEMMA If τ is the diagonal word o a labelled Dyck path iff

(1) $s_i(\tau) \neq 0 \quad \forall i$

(2) $\{s_i(\tau) \mid i \in [n]\}$ is an <u>interval</u>

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$$= \sum_{\substack{\tau \in \mathfrak{S}_n^{\bullet k} \\ \mathrm{sched}(\tau) = 1^n}} t^{\mathrm{revmaj}(D)}$$

$\mathfrak{S}_n^{\bullet k}$ → permutations with k decorations

A PERMUTATION BIJECTION

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$S_n^{\bullet}(1^n)$

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THM (Corteel, Josuat-Vergès, VW)

there exists a bijection $\phi : \mathfrak{S}_n^{\bullet}(1^n) \longrightarrow \mathfrak{S}_n$ such that

(1) $\mathrm{revmaj}(\tau) = \mathrm{inv}_3(\phi(\tau))$

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We construct this bijection by means of isomorphic generating trees.

GENERATING TREE FOR DECORATED, SCHED 1^n PERMUTATIONS AND REMAJ

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CONVENTION: $\sigma_0 = 0$

DEF Given $\sigma \in \mathfrak{S}_n^{\bullet}(1^n)$ and $k \in \{1, \dots, n+1\}$

$$\sigma = \overset{0}{\overset{\shortparallel}{\sigma_0}} \overset{(\bullet)}{\sigma_1} \cdots \overset{(\bullet)}{\sigma_n} \quad \xrightarrow{+k}$$

$$k + \sigma = 0 \; \underbrace{\overset{*}{k} \, (\overset{(\bullet)}{\sigma_1} + k) \cdots (\overset{(\bullet)}{\sigma_n} + k)}_{\mathrm{mod}(n+1)}$$

* Decorate k

$\iff k > n+1 -$ first undecorated letter

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EXAMPLE $\sigma = 0 \; 1 \; 4 \; 2 \; 3 \; 6 \; 5 \; 7$

$3+\sigma = 0 \; 3 \; 4 \; 7 \; 5 \; 6 \; 1 \; 8 \; 2$

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EXAMPLE $\sigma = 0\ 1\ 4\ 2\ 3\ 6\ 5\ 7$

$3+\sigma = 0\ 3\ 4\ 7\ 5\ 6\ 1\ 8\ 2$

We generate all decorated schedule 1^n permutations by starting from $1 \in \mathfrak{S}_1^\bullet(1^n)$ and generating the descendants $1+\sigma, \dots, n+1+\sigma$ at each level of the tree

GENERATING TREE FOR DECORATED, SCHED 1^n PERMUTATIONS AND REVMAJ

1

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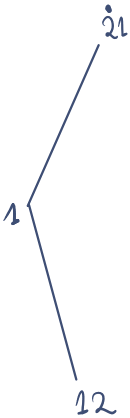

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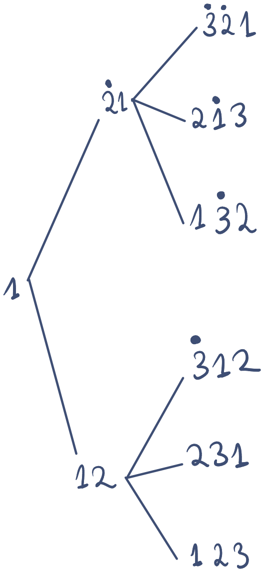

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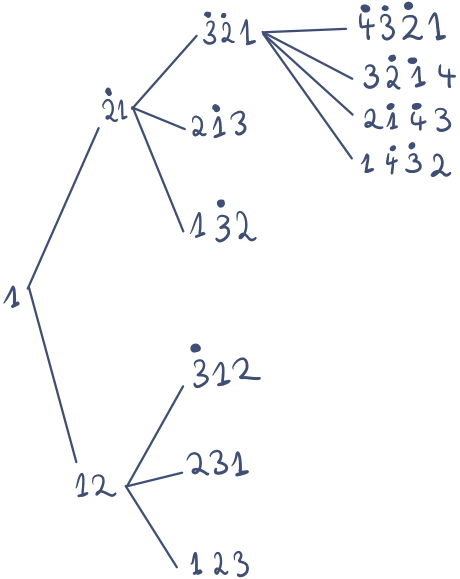

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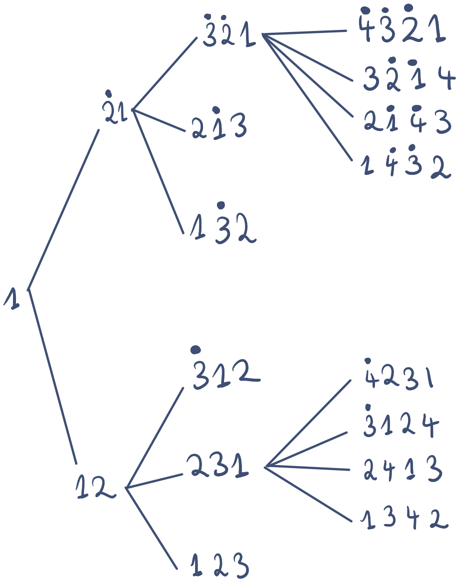

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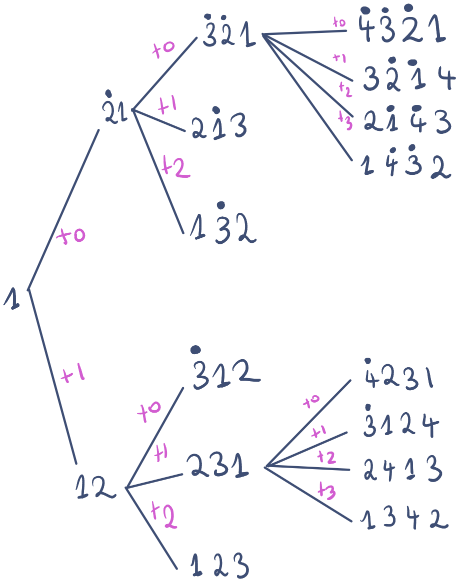

remaj

THE GENERATING TREE FOR PERMUTATIONS AND INV_3

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Given $\sigma \in \mathfrak{S}_n$, its $n+1$ descendants will be obtained by adding a letter $k \in \{1, \ldots, n+1\}$ at the end and augmenting by 1 all the letters of σ that are $\geq k$.

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EXAMPLE $k = 4$

$$6\ 8\ 3\ 1\ 4\ 5\ 7\ 2$$

$$\downarrow$$

$$7\ 9\ 4\ 1\ 5\ 6\ 8\ 2\ 3$$

THE GENERATING TREE FOR PERMUTATIONS AND inv_3

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EXAMPLE $k=4$ 6 8 3 1 4 5 7 2

$$\downarrow$$

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What is the effect on the inv_3?

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$$\downarrow$$
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THE GENERATING TREE FOR PERMUTATIONS AND INV_3

1

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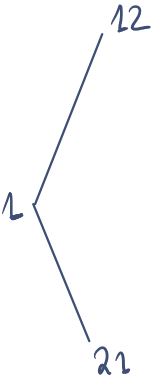

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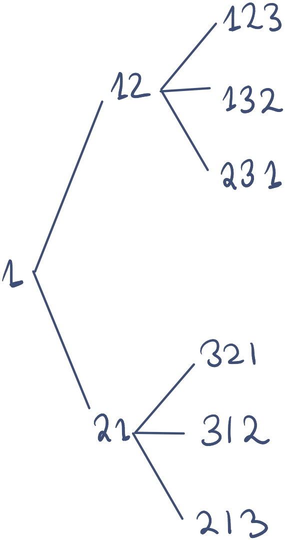

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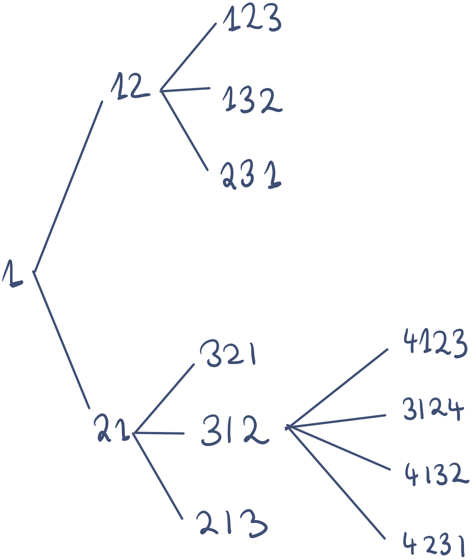

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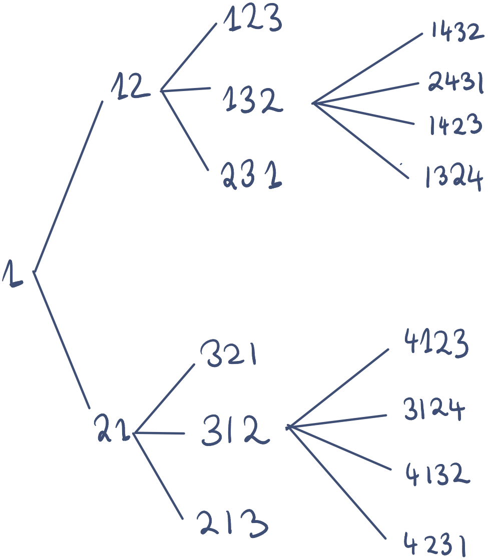

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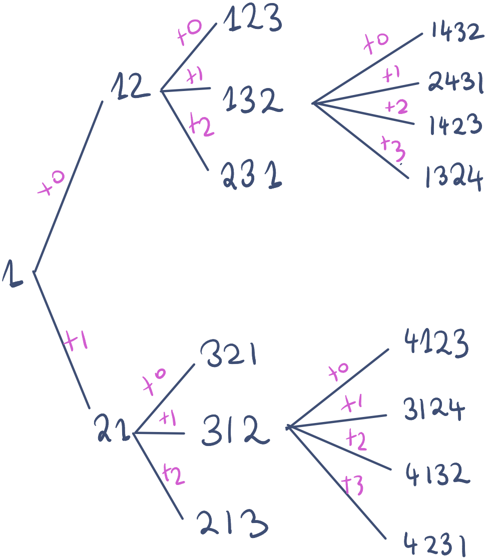

inv_3

Future directions

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- $\langle \nabla e_n, h_\lambda \rangle|_{q=-1}$ for $\lambda \neq 1^n$

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- Other cool symmetric functions where we observed nice $q=-1$ behavior

$$\nabla \omega(p_n) \qquad \nabla E_{n,k} \qquad \Theta_{e_k} \Theta_{e_l} \nabla e_{n-k-l}$$

Thank you very much for
your attention